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Courant Institute of
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Magneto-Fluid Dynamics Division

# Singular and Nonuniform Limits of Solutions of the Boltzmann Equation

Harold Grad

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# Singular and Nonuniform Limits of Solutions of the Boltzmann Equation

HAROLD GRAD

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## Abstract

A survey is made of various singular and nonuniform limits of solutions of the Boltzmann equation. Included are the limits of large and small mean free path, fast and slow molecules, initial, boundary, and shock layers, and singular features of the collision integral. Several new results are given including an analysis of the Hilbert expansion in steady flow, of the kinetic boundary layer including a complete formal theory of slip coefficients, and of the profile of a very strong shock.

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## 1. Introduction

The study of singularities and nonuniform limits appears to be a major preoccupation of applied mathematicians. Aside from the intrinsic interest of such problems, there is good reason for this concentration. A good existence theorem will include as a corollary continuous or smooth dependence of the solution on parameters. If a parameter which occurs in the problem has a natural range of variation, then one can expect the behavior at the endpoints to be singular; otherwise the solution could be continued. Thus a detailed description of the singular endpoints, together with continuous dependence in between, should give a uniformly valid qualitative picture of the entire problem.

There is also a quantitative profit that can be gained from knowledge of the singular endpoints. It is frequently possible to "subtract off" the singular part and then obtain quite accurate answers for the remainder using relatively unsophisticated or ad hoc procedures. In kinetic theory, this combination of relatively recent precise investigations of many special limiting cases with the older moment and polynomial approximations offers the liklihood of quite accurate practical calculations even if convergence is not yet established.

In a previous talk at this Symposium*, existence theorems covering a certain area of kinetic theory were given together with a rigorous analysis of the behavior at one singular limit (small mean free path). We now survey all (or at least many) of the singular limits to which solutions of the Boltzmann equation fall prey. Since the available existence theory does not cover this extended territory, the present treatment will be formal rather than rigorous.

For coherence in the oral presentation, the emphasis was placed on a broad survey rather than on the new material. We reverse this emphasis in the written text since there are other sources available for the details of many of the nonuniform limits which are here coordinated.** The new material consists of a unified formal treatment of the small mean free path limit in steady boundary value problems (Secs. 7-10) and an analysis of the profile of a very strong shock (Sec. 11).

The boundary value problem separates into an analysis of the interior and an evaluation of asymptotic boundary conditions. For the interior analysis, an asymptotic

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*"Asymptotic Equivalence of the Navier-Stokes and Nonlinear Boltzmann Equations," New York, April 22, 1964; published in Proceedings of Symposia on Applied Mathematics (American Mathematical Society, 1965), Vol. XVII.

**References will be given as appropriate.

procedure is given which is superior to the direct Hilbert expansion or the conventional Chapman-Enskog expansion; (but in special cases this reduces to a degenerate form of the Hilbert expansion). This analysis of the interior of the domain is supplemented by a formally complete description of the slip flow boundary conditions in a general domain. Numerical evaluation of the slip coefficients requires solution of certain special one-dimensional problems generalizing the Kramers problem.

The strong (strictly speaking, infinite strength) shock profile is reduced to a relatively nonsingular problem by introducing the correct scaling in physical space and in velocity space after subtracting off a suitable singularity. This treatment can be qualitatively interpreted as a more rational refinement of the ad hoc Mott-Smith bimodal approximation. In the correct scaling, it appears that a very simple interpolation can be made between the infinitely strong and weak shock profiles.

A survey of various singular limiting cases is presented in Secs. 2-6, followed by more detailed consideration of steady slip flow in Secs. 7-10 and of the shock layer in Sec. 11. We conclude in Sec. 12 with some opinions on future trends in kinetic theory.

We should like to acknowledge many useful discussions with Dr. Yoshio Sone in connection with the material in Sec. 10.

## 2. Nonuniform Limits

The argument of the Boltzmann equation is a distribution function, $f(\xi,x,t)$. This represents the density at a given instant in the phase space of a molecule; $x$ = position and $\xi$ = velocity. We find nonuniform limits at each extreme of every argument of $f$. There are nonuniform limits as $x \longrightarrow 0$ (boundary layer) and as $x \longrightarrow \infty$; also as $t \longrightarrow 0$ (initial layer) and as $t \longrightarrow \infty$ (approach to equilibrium); also for slow molecules, $\xi \longrightarrow 0$, and fast molecules, $\xi \longrightarrow \infty$. If dimensionless variables are introduced for $\xi$, $x$, and $t$ ($\xi$ relative to a representative mean thermal speed, $x$ and $t$ relative to some given scale)*, then a single dimensionless parameter, $\varepsilon$, appears in the Boltzmann equation. This is the ratio of mean free path to the given scale length or, equivalently, of the mean collision time to the scale time. Both limits, $\varepsilon \longrightarrow 0$ (continuum) and $\varepsilon \longrightarrow \infty$ (free flow) are singular.

Since macroscopic fluid dynamics occurs as a limiting case, $\varepsilon \longrightarrow 0$, in the theory of the Boltzmann equation, all singular and nonuniform limits of classical fluid dynamics are automatically included. These encompass an assortment of boundary layers, wakes, shock layers, various linearizations, behavior at infinity, small and large Mach number

*We prevent the appearance of a Mach number in the equation by taking the $\xi$ scale equal to the ratio of the $x$ and $t$ scales.

and Reynolds number, and many more. The singular approach to fluid dynamics as $\varepsilon \to 0$ is, of course, supplemental to the strictly fluid nonuniformities.

An interesting point is the fact that the parameter $\varepsilon$ which gives rise to fluid dynamics in the limit $\varepsilon \to 0$ occurs within fluid dynamics proper as a finite parameter, $\varepsilon = M/Re$ (Knudsen number equals Mach over Reynolds number). From fluid dynamics alone we could not suspect that finite (i.e., not small) values of $M/Re$ are intrinsically unreliable. The limit $\varepsilon \to 0$ is singular in both fluid dynamics and kinetic theory, with the two sets of complications superposed. For example, a fluid dynamic boundary layer expansion in $\varepsilon$ ($Re \to \infty$ at finite M) requires supplementation by kinetic effects, usually at second order in $\varepsilon$.

The fact that fluid dynamics emerges from the Boltzmann equation in a certain limit is intimately tied to the laws of conservation of mass, momentum, and energy. The conservation laws result from certain elementary properties of the collision integral, viz. that the nullspace is nontrivial (and finite dimensional). This situation can be compared to the case of neutron transport where there is a similar equation, also called the Boltzmann equation.*

*Cf. N. Corngold, "Some Recent Results in the Theory of the Transport of Thermal Neutrons," to appear in Symposia in Applied Mathematics (American Mathematical Society), Vol. XX.

In the neutron case, conservation does not hold. As a result solutions will, in general, either grow exponentially or decay exponentially on the time scale of the mean collision time. To identify the borderline case between growth and decay (e.g., in its dependence on geometry) is one of the basic problems in neutron transport. In a sense, the Boltzmann equation for a gas (or Boltzmann's equation) represents a set of measure zero from the point of view of neutron transport. But it is exactly this borderline quality which gives a gas its unique and distinctive features (including fluid flow) and makes a significant portion of the mathematical analysis much more delicate.

Our primary concern is with nonuniformities which are peculiar to kinetic theory (one of which is the singular connection between the fluid and kinetic descriptions, cf. Sec. 5). Many nonuniformities arise from the specific form of the kernel of the collision term of the Boltzmann equation which represents the nature of the intermolecular force. It is important to realize that "the" Boltzmann equation is not a specific equation but a generic term for a class of equations (similar in extent to the term integral equation). There are various types of collision kernels, some more singular than others. In particular, one feature that can be extracted from the collision kernel is the

velocity-dependent collision frequency $\nu(\xi)$ and the related velocity-dependent mean free path $\lambda(\xi) = \xi/\nu(\xi)$. Much of the singular behavior for small and large $\xi$ can be related to the corresponding limiting behavior of $\nu(\xi)$ and $\lambda(\xi)$, cf. Secs. 3 and 4.

One class of collision kernels about which almost nothing is known (in a basic sense the most important class) is that for a non-cutoff intermolecular potential.* In this case no simple collision frequency $\nu(\xi)$ or mean free path $\lambda(\xi)$ can be defined, and the nature of the collision operator (e.g., whether it is an integral operator, singular integral operator, differential operator, etc.) is not known.** For a certain type of cutoff (in angle of deflection), a relatively complete mathematical theory has been developed [Grad, 1963b]. For another cutoff (intermolecular distance) a few results have been obtained.*** One would like to know, on physical grounds, that "many" properties of solutions of the Boltzmann equation are insensitive to the type of cutoff and are, indeed, similar

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*For a result in a special case see [Smolderen, 1965; Willis, 1965].

**An approach to this problem is described in [Finkelstein, 1965].

***C. Cercignani, private communication.

to those for non-cutoff potentials. The one case in which this is known to be true is in the formula for the calculation of transport coefficients [Burnett, 1936]. What is probably the case more generally is that the precise nature of certain mathematical singularities is sensitively dependent on the nature of the cutoff (or on the exact decay of a non-cutoff potential), even though some gross properties of the gas flow will not be so sensitive.

We have mentioned that all the singular limits of classical fluid dynamics are contained within kinetic theory. This is not quite so; some of the bad behavior may be wiped out. In particular, nonlinear and linearized results are related quite differently in the fluid and kinetic versions. The streaming term of the Boltzmann equation is strictly linear; nonlinearity enters only through the quadratic collision term. But fluid equations are obtained from the Boltzmann equation by a procedure which operates only on the linear streaming terms, explicitly ignoring the collision term; the relevant moments of the collision term vanish because of conservation. The complete set of fluid conservation equations is therefore linear (in the proper variables) until it is "simplified" by use of the strongly nonlinear Maxwellian distribution function. The nonlinearity of the fluid equations (and the associated equations of state) is imposed by the nonlinear Boltzmann collision term, but only in a rather devious manner.

To illustrate this divergence between the two theories, consider the flow at finite Mach number past a small sphere or cylinder (small compared to a mean free path). The fluid solution is definitely nonlinear regardless of the size of the body. The kinetic solution also yields a strong disturbance near the body. But this strong disturbance is governed by the strictly linear streaming terms of the Boltzmann equation; the collision term is negligible until about a mean free path away, at which distance the disturbance has become small. The limiting flow is given as a finite amplitude solution of a linear equation (free flow). The leading term of the departure from free flow is nonlinear in three dimensions but is linear in two dimensions.

A similar example is given by high frequency sound, e.g., as generated by an oscillating wall.* The conventional fluid linearization which takes derivatives as well as fluid perturbations to be small becomes increasingly poor as the frequency and wave number become large. But in the kinetic analysis, a disturbance with frequency which is large compared to the collision frequency is dominated by the streaming terms, and the nonlinear collision

*For the solution with a relaxation model see [Weitzner, 1965] and [Mason, 1965]; with the Boltzmann equation at high frequency see [Grad, 1966].

term offers only a minor correction to the behavior of even a finite amplitude wave. For a fixed small amplitude of the disturbance, linearization of the Boltzmann equation will be most unreliable at some intermediate frequency.

A similar conclusion is reached for the initial value problem. For the Boltzmann equation, the natural time scale is a mean collision time. One would expect the discrepancy between a linear approximation and the correct nonlinear solution to grow with time on this time scale. But if the disturbance is very long compared to the mean free path, a much longer (macroscopic) time scale emerges. These expectations are verified by quantitative estimates in linear and nonlinear existence theorems [Grad, 1965]. Thus the worst case with regard to linearization is, again, the intermediate region with wave length comparable to (or probably somewhat larger than) the mean free path.

The Navier-Stokes level of the Chapman-Enskog expansion amounts to a linearization in velocity space (small deviation from local Maxwellian) in a problem which may be strongly nonlinear in physical space. But even the linearization in velocity space is not uniformly valid for large velocity [Narasimha, 1967] (as evidenced by the distribution function becoming negative). Even in a problem in which the fluid disturbance is small, i.e.,

linear in physical space, the correct treatment of the high velocity tail is always nonlinear; the tail of the distribution, although small, is not small relative to a Maxwellian.

Nonlinearity can even introduce a dissipative mechanism where there is none linearly. In the limit of small mean free path, $\varepsilon$, one aspect of dissipation occurs rapidly on a time scale $\varepsilon$, during which the Boltzmann H-function becomes approximately equal to the thermodynamic entropy S; this is then followed by a much slower viscous dissipation on a time scale $1/\varepsilon$. The linear problem is non-dissipative on the intermediate time scale $O(1)$. But in a nonlinear problem, shock waves provide additional dissipation (which involves H and S differently) on the time scale $O(1)$.

The moment and polynomial approximations which were originally introduced to break away from the fluid regime [Grad, 1949] are essentially ad hoc. They usually supply fairly good approximations in a low order approximation, but no rapid improvements are found even at considerably extra labor [Pekeris, et al., 1962]. But, if an explicit singularity or nonuniformity which prevents rapid convergence is recognized, the polynomial expansion can be altered to suit, and, although still ad hoc, the level of accuracy may be greatly improved even at low order. The most elementary example is the inclusion of

two-sidedness in the distribution by using half range expansions [Gross and Ziering, 1958; Lees 1959].

Another example is given by the fundamental solution in steady flow. For a relaxation model of the Boltzmann equation the Fourier transform of the solution is explicit [Grad, 1959]. For the actual Boltzmann equation we are not so fortunate. However, the singularity near the source is explicit to a certain number of terms in an expansion. Subtracting this off leaves a residue which is bounded and can be expected to be well-approximated by a simple moment expansion (Euler, Navier-Stokes, or Thirteen Moment). In addition, the leading term at infinity is governed by fluid dynamics and is therefore given correctly by a moment approximation. This scheme has been carried out [Yu, 1967]; unfortunately the singularity at the origin, although "explicit", involves very complex high-dimensional integrals.

A more successful application of this technique is to the strong shock. We recall that for a Navier-Stokes profile, proper scaling (viz. introducing the downstream mean free path as a reference length) gives a shock thickness which does not vary by more than a factor two or three and a profile which is not very far from universal as the strength varies from zero to infinity [Grad, 1952]. We now find for the Boltzmann equation, after subtracting

off an explicit singularity in velocity space, that there is a definite limiting profile in $\xi$ as well as x for the distribution function at infinite strength. This limiting distribution function can be approximated by relatively crude methods (e.g., by a local Maxwellian, which gives no result at all if the singularity is not removed). One conclusion is that the shock profile can be qualitatively approximated over the entire range from zero to infinite strength by simple interpolation (Sec. 11).

This very brief summary of singular limits is continued with a slightly more elaborate discussion of the nonuniformities which are introduced by the collision kernel and at the large mean free path limit in the next two sections.

## 3. The Spectrum--Fast and Slow Molecules

With the exception of the finite nullspace of the collision operator (which expresses conservation and leads to fluid dynamics), probably the single most important qualitative feature of the linear Boltzmann equation is the almost universal occurrence of a continuous spectrum. There are two sources of a continuum, the streaming operator, $\xi \cdot \nabla$, and the collision operator L.

The streaming operator alone, as in

$$\frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial x} = 0 \ , \tag{3.1}$$

has the entire imaginary axis as a continuum. This leads to the decay in time of an arbitrary initial function $f(\xi,x,0)$ [Grad, 1961] and the decay in space of the signal induced by an oscillating wall in a collisionless gas [Kahn and Mintzer, 1965]. It is also the dominant effect at high frequency even in the presence of collisions.

The collision operator by itself,

$$\frac{\partial f}{\partial t} + L(f) = 0 \tag{3.2}$$

has a continuum covering part of the real axis for most molecular force laws which have been investigated up to the present time [Grad, 1963b].

Since the operator L is symmetric and $\xi \cdot \nabla$ is antisymmetric, and both operators are unbounded, the spectrum

of the sum, $\xi \cdot \nabla + L$, is not evident. There are a large number of interesting perturbation problems with a few proved results and many plausible conjectures. It is relatively easy to make reliable plausible estimates of the continuous spectrum. This yields valuable and detailed qualitative information about solutions of the Boltzmann equation. It is also the most important single criterion for assessing what phenomena are likely to be misrepresented by a simplified relaxation model of the Boltzmann equation. In particular, many singular features are immediately visible from a study of the continuous spectrum.

For the initial value problem (3.2), the continuum is the set of values taken by the collision frequency $\nu(\xi)$. For a class of potentials called <u>hard</u> we have $\nu(0) = 1$ increasing to $\nu(\infty) = \infty$; for a <u>soft</u> potential, $\nu(0) = 1$ decreases to $\nu(\infty) = 0$. In a gas of hard molecules the high velocity tail of an initial distribution $f(\xi,0)$ will instantly equilibrate, this effect gradually penetrating towards the center. On the other hand, a soft gas will reach equilibrium in the center of the distribution in a time $O(1)$, and this will spread more and more slowly to the tail.

A more interesting illustration is given by a boundary value problem,

$$\xi_1 \frac{\partial f}{\partial x_1} + L(f) = 0 \qquad (3.3)$$

The continuous spectrum (plausible conjecture) is in this case determined as the set of values taken by the reciprocal of the free path, $1/\lambda_1(\xi) = \nu(\xi)/\xi_1$. In most cases the continuum covers the entire real axis. Because $\lambda_1 = 0$ for $\xi_1 = 0$, there is nonuniform convergence in velocity space for the distribution function near a wall. The difference between $f(\xi,x)$ and the emitted wall distribution $f^+(\xi,0)$ $(\xi_1 > 0)$ does not approach zero in a velocity boundary layer of thickness $\xi_1 = O(x)$. As a consequence, certain moments (e.g., temperature, tangential velocity) have infinite slopes at the wall.*

At the opposite extreme we have $\lambda_1 \to \infty$ for $\xi \to \infty$ (with the single exception of hard spheres). In this case, fast particles will exert an influence at long distances. It has been found, for example, that the weak shock profile has a nonexponential decay at infinity, apparently contradicting the Navier-Stokes result [Lyubarski, 1961]. The reason is the large penetration of fast particles past the bulk of the shock transition. There is an inversion of limits in that the Navier-Stokes profile is

*For a relaxation model see [Sone, 1964a,b]; for the Boltzmann equation see [Grad, 1966].

correct for a relatively greater portion of the profile as the shock strength decreases. The effect is small because there are exponentially few fast particles. Whether this effect is important in a strong shock has not yet been determined.

In the presence of a wall oscillating at a fixed frequency,

$$i\omega f + \xi_1 \frac{\partial f}{\partial x_1} + L(f) = 0 \qquad (3.4)$$

the spectrum [i.e., of $\partial f/\partial x_1 = \mathcal{L}(f)$] moves into the complex plane. Many qualitative features of the solutions can be predicted just from a study of the continuous spectrum, e.g., the existence for large $\omega$ of sub-boundary layers in the approach of f to the wall distribution $f^+$, and the disappearance of a sound wave (i.e., of all discrete normal modes) for sufficiently high $\omega$.

The situation is qualitatively similar (e.g., with regard to the complex continuous spectrum swallowing up the normal modes at small wavelength) for an initial value problem at fixed wavelength,

$$\frac{\partial f}{\partial t} + i\xi_1 k_1 f + L(f) = 0 \qquad (3.5)$$

A more detailed description of these spectra is discussed in [Grad 1964a, 1964b, 1966].

## 4. Large Mean Free Path--The Ubiquitous Logarithm

In the limit of large mean free path we write $\varepsilon = K$ (Knudsen number). There are two distinct reasons why $K \longrightarrow \infty$ does not approach uniformly to the free flow limit. One reason is that $\lambda(\xi) \longrightarrow 0$ for small $\xi$ so there is always a class of slow particles with small free path no matter how large K may be. A second reason is that in an exterior or otherwise unbounded domain there will always be distant collisions.

There is a simple mathematical lemma which, although mathematically trivial, is responsible for many important phenomena which are physically profound and subtle, including both of the above mentioned sources of nonuniformity at large K. The divergent integral $\int dr/r$ can be given a finite value by introducing a convergence factor

$$I(\varepsilon) = \int_1^\infty \phi(\varepsilon r)\, \frac{dr}{r} \tag{4.1}$$

(the divergence near $r = 0$, which we dismiss with the finite limit $r = 1$, can be studied in exactly the same way). We assume that $\phi(0) = 1$ and $\phi(s)$ is smooth near $s = 0$; also that $\phi$ is integrable at infinity. Writing

$$I = \int_{\varepsilon}^{\infty} \phi(s)\,\frac{ds}{s} = \int_{\varepsilon}^{1} (\phi-1)\,\frac{ds}{s} + \int_{\varepsilon}^{1} \frac{ds}{s} + \int_{1}^{\infty} \phi\,\frac{ds}{s} \qquad (4.2)$$

we see that the first and third integrals are finite as $\varepsilon \longrightarrow 0$. Consequently,

$$I(\varepsilon) = \log 1/\varepsilon + O(1) \; . \qquad (4.3)$$

We observe that the dominant term $\log 1/\varepsilon$ is completely independent of the nature of the convergence factor, depending only on the length scale $\varepsilon$; e.g., its value would be the same if $\varepsilon$ were replaced by $2\varepsilon$. Also the dominant contribution to the integral comes from values of $r$ which are large compared to the lower limit, $r = 1$, but are small compared to $r = 1/\varepsilon$ [where $\phi$ deviates from the value $\phi(0) = 1$]. The error term, $O(1)$ in (4.3), does depend on the entire function $\phi(s)$. In other words, the leading term of an expansion of $I(\varepsilon)$ is local and extremely insensitive, whereas higher order terms are global and more complicated.

The special choice

$$\begin{aligned} \phi(s) &= 1 \; , \quad 0 < s < a \\ \phi(s) &= 0 \; , \quad s > a \end{aligned} \qquad (4.4)$$

which amounts to cutting off the simple integral $\int dr/r$ at $r = a/\varepsilon$, also gives a leading term which is independent of the cutoff parameter a.

An integral of the form (4.1) arises commonly when a problem presents two length scales, as in a boundary layer analysis. More generally,

$$J(\varepsilon) = \int_1^\infty \phi(\varepsilon r) \frac{dr}{r^n} \tag{4.5}$$

can be expanded as

$$J = \frac{\phi(0)}{n-1} + \frac{\phi'(0)}{(n-1)(n-2)} \varepsilon + \ldots + \frac{\phi^{(n-1)}(0)}{(n-1)!} \varepsilon^{n-1}[\log \frac{1}{\varepsilon} + O(1)] \tag{4.6}$$

There is a polynomial in $\varepsilon$ ending with the appearance of a logarithm. All the listed coefficients are local [depending on derivatives of $\phi$ at $r = 0$]; higher order terms in the expansion depend globally on $\phi$.

Similar expansions can be made for

$$K(\varepsilon) = \int_1^\infty \phi(\varepsilon r)\psi(r)dr \tag{4.7}$$

in terms of as many moments of $\psi(r)$ as are finite and an equivalent number of derivatives of $\phi$ at the origin. The

boundary layer interpretation of this result states that the dominant part of the expansion of $K(\varepsilon)$ (up to and including the logarithm) can be evaluated locally, within the boundary layer, without the necessity of matching to the exterior region.

A common tool in investigating solutions of the Boltzmann equation is to write it as an integral equation. In the integrand will appear a factor

$$\exp[-\nu(\xi)x/\xi_1] = \exp[-x/\lambda(\xi)] \ . \tag{4.8}$$

Near a wall, $x \sim 0$, we can expand the exponent in powers of x except for small $\xi_1$ where $\lambda \sim 0$. Integrals with respect to $\xi$ of such an expression will yield a leading term $x \log 1/x$ (or higher order terms $x + x^2 + \ldots + x^r \log 1/x$ for moments which have powers of $\xi_1$ as a factor).

For large mean free path, the relevant factor appears as

$$\exp[-x/K\lambda(\xi)] \tag{4.9}$$

With x fixed and nonzero, this "convergence factor" is regular in $1/K$ except where $\lambda \sim 0$. Again we obtain $(1/K)\log K$ as the leading term for an integral over this

expression. But this is correct only in one dimension. At a point definitely removed from a wall, the dimension of the manifold on which the appropriately defined $\lambda(\xi)$ vanishes depends on the dimensionality of the problem. In one dimension (e.g., two parallel flat plates), $\lambda = 0$ when $\xi_1 = 0$ and $(\xi_2, \xi_3)$ are arbitrary. In two dimensions (e.g., two concentric cylinders), $\lambda = 0$ when $\xi_r = \xi_\theta = 0$ and $\xi_z$ is arbitrary. In three dimensions (e.g., two concentric spheres, or any bounded interior domain), $\lambda = 0$ only when the vector $\xi = 0$. Therefore, in two dimensions there will be a term $1/K$ followed by $(\log K)/K^2$, and in three dimensions two terms, $1/K + 1/K^2$, followed by $(\log K)/K^3$. Furthermore, the evaluation of these coefficients is local, depending essentially on the free flow distribution, whereas any higher order terms pose difficulties of matching in velocity space.

In an external domain we have, in addition to the slow particles treated above, the effect of distant particles. The same integral (4.9) appears as a convergence factor, now integrated with respect to x from infinity instead of from a wall. In two dimensions (e.g., flow past a cylinder), the function which is multiplied by this exponential convergence factor decays as $1/x$ (this represents the free flow disturbance which is proportional to the angle subtended by the object at a distance x). Thus the leading

term is log K. An additional factor 1/K is explicit in the integral which is evaluated; thus the modification of the free flow drag is (log K)/K. Similarly, in three dimensions where the influence function decays as $1/x^2$, we obtain $1/K + (\log K)/K^2$ as the leading terms [after free flow which is scaled to be O(1)].

In an exterior domain where both the slow and distant particle nonuniformities are present (arising from integration over $\xi$ and x respectively), it is the distant particle effect which dominates, always entering at lower order. Alternatively, the logarithm enters at a lower order in a two-dimensional exterior problem, say drag past a cylinder, than in a two-dimensional interior problem, such as drag between two concentric cylinders.

We have been intentionally vague in not stating above exactly what integral is being evaluated. The descriptions given are basic in the sense that there exists some interesting macroscopic quantity with the logarithm appearing at the given order. Other observables, involving integrals over functions which vanish at small velocity or involving some special symmetry, may exhibit a logarithm only later on in the expansion. For example, in linearized one-dimensional heat flow, although the temperature gradient at the midplane is (log K)/K [Willis, 1958], the ratio of total heat flow

to the difference in wall temperature (effective conductivity) is 1+1/K (because the heat flow is a moment of f with a factor $\xi_1$). But in a nonlinear problem with a finite temperature difference, the effective conductivity (normalized at fixed total mass in the system) will behave like 1 +(log K)/K because of a loss of symmetry about the midplane.

The integrals which arise in this subject will, in general, involve combinations of boundary layers $x \sim 0$, $\xi \sim 0$ and nonuniformities $x \sim \infty$, $K \sim \infty$. Although the basic concepts and mathematical analysis of these linked nonuniformities are elementary, the bookkeeping can be quite complicated, involving the number of dimensions, the degree of symmetry, and the specific observable. A complete list has not been written down, although many specific examples have been studied, e.g. [Willis, 1958; Cercignani, 1962; Su, 1964; Sone, 1964; Pao, 1965; Grad, 1966; Yu, 1967].

It is interesting to note that the same mathematical lemma, and approximately the same physical situation, occurs in the derivation of a kinetic equation. To lowest order in the appropriate expansion parameter through which the Boltzmann equation arises, there is no formal difficulty. To higher order, including dense gas corrections, one finds a nonuniformity arising from slow particles and another from

distant particles [Grad, 1958; Frieman and Goldman, 1966, and many others].

Essentially the same mathematical situation also arises in evaluating transport coefficients in a plasma. A formal divergence, arising from the long range Coulomb potential, is eliminated on introducing the concept of a shielded potential [Spitzer, 1956]. Without performing any of the lengthy computations, it is immediately evident that the dominant term of the electrical conductivity (i.e., the Coulomb logarithm) must arise with identical numerical coefficients whether we employ the Fokker-Planck equation or the Boltzmann equation, and whether we use a spherical Debye shielding cloud, or a more sophisticated distorted shielding cloud, or whether we simply cut off the divergence at the Debye length [Grad, 1962]. The only fact that we must know is the scale length of the shielding, without regard to the details of the shielding process. But to carry the computation beyond the dominant term is formidable and requires a correct model for the shielding.

## 5. Small Mean Free Path--Introduction

The limit of small mean free path probably represents the most extreme of all nonuniformities to be found within the scope of the Boltzmann equation, but it is also the best understood and documented. In this limit, the gas description is reduced from a distribution function $f(\xi,x,t)$ to the macroscopic fluid coordinates $\rho(x,t)$, $T(x,t)$, $u(x,t)$.

The Boltzmann equation can be written in skeleton form as

$$\frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial x} = Q(f,f) \tag{5.1}$$

where Q is a quadratic operator on f which vanishes when applied to a local Maxwellian

$$f_M = \frac{\rho}{(2\pi RT)^{3/2}} \exp\left[ - \frac{(\xi-u)^2}{2RT} \right] . \tag{5.2}$$

By local we mean that $\rho$, T, and u may be arbitrary functions of x and t. Under appropriate scaling, the Boltzmann equation becomes

$$\frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial x} = \frac{1}{\varepsilon} Q(f,f) \tag{5.3}$$

or in a linearized version

$$\frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial x} + \frac{1}{\varepsilon} L(f) = 0 \tag{5.4}$$

The nonlinear form can be obtained by the substitution $f \longrightarrow f/\varepsilon$ in (5.1) and the linear version by the substitution $f \longrightarrow \frac{1}{\varepsilon} f_M + f$ where $f_M$ is here an absolute Maxwellian with constant parameters $\rho$, T, u. The significance of the substitution $f \longrightarrow f/\varepsilon$ is that for small $\varepsilon$ the density is large and therefore the mean free path is small. In the linearization, $\varepsilon$ represents the mean free path of the unperturbed constant state, $f_M/\varepsilon$.

It is evident that the limit $\varepsilon \longrightarrow 0$ is singular since all derivatives are lost. For small $\varepsilon$ we should expect to have $Q(f,f) \sim 0$ and $f \sim f_M$ except in regions where $\partial/\partial t$ or $\partial/\partial x$ become unbounded. To be more definite, take a fixed domain with appropriate fixed boundary and initial conditions on f, independent of the parameter $\varepsilon$. The qualitative behavior for small $\varepsilon$ of the solution $f(\xi,x,t;\ \varepsilon)$ is described by a conjecture [Grad, 1958] which states that f approximates $f_M$, and the fluid parameters in $f_M$ obey fluid equations, except where this is evidently impossible.

Specifically, there are three locations where f cannot approximate $f_M$ closely:

1) in an initial layer, since the initial value $f(\xi,x,0)$ is arbitrary;

2) in a boundary layer where the "half" distribution of molecules leaving the wall is, in principle, arbitrary;

3) in a shock layer where one cannot conclude that $Q(f,f) \sim 0$ since $\partial/\partial t$ and $\partial/\partial x$ are of order $1/\varepsilon$ (the shock thickness scales as $\varepsilon$).

The conjecture states that, except in these three layers of nonuniformity, the Hilbert or Chapman-Enskog formal expansions in $\varepsilon$ are asymptotic to the actual solution $f(\xi,x,t;\ \varepsilon)$. This conjecture has been promoted to a proved theorem in linear and weakly nonlinear formulations of the initial value problem [Grad, 1963a,b; 1965].

Accepting the conjecture more generally, the question immediately arises of matching the asymptotic expansions across the three types of boundary layers within which they are not valid. The fluid equations (which can take several forms) yield solutions only when they are completed with initial data, boundary data, and possibly shock jump conditions. The relation between the given Boltzmann boundary data (half distribution) and the proper asymptotic fluid boundary data requires analysis of a boundary layer, and

similarly for the initial and shock layers. A crude analysis of the boundary layer was given by Maxwell and led to the concept of boundary slip (of tangential velocity and temperature). In the same terminology we can talk of initial slip and shock slip. The initial slip represents the amount by which the true initial values of the fluid state (given as moments of the assigned initial distribution function) must be altered so that, when used as initial values in fluid equations, they give rise to the correct future fluid state as it would be found by solving the Boltzmann equation. Similarly, the shock slip represents the correction to the classical Hugoniot condition which is required for a fluid calculation to yield the same result as solution of the Boltzmann equation. A related concept of shock slip has also been found within fluid dynamics proper as a modification of the inviscid Hugoniot conditions to take into account the interaction between a shock and a boundary layer [Probstein and Pan, 1963]. Although viscous fluid equations can presumably give a smooth shock transition, to approximate the kinetic solution correctly, they must be allowed to have discontinuities.

The small mean free path limit has been split into two problems. The first, in the interior (away from initial, boundary, and shock layers), is solved by a

Hilbert or Chapman-Enskog expansion (except, as we shall see later, when they require modification). The second consists in the analysis of three kinetic boundary layers in order to complete the fluid formulation with initial, boundary, and jump conditions. Note that the type of boundary data that is required for inviscid fluid equations is quite different from what is required for viscous equations. The kinetic boundary layer analysis will therefore differ depending on what fluid model is selected.

## 6. The Initial Value Problem

The solution to the problem of initial slip is essentially complete [Grad, 1963; 1965]. For the nonlinear problem the solution is formal; an algorithm is given for the computation of the asymptotic initial fluid state in terms of the given initial distribution function. For the linear and weakly nonlinear problems, the solution $f(\xi,x,t;\ \varepsilon)$ has been shown to exist, and the formal expansions in $\varepsilon$ have been proved to be asymptotic to this solution.

To be more precise, let $f(\xi,x,t;\ \varepsilon)$ be the solution of the Boltzmann equation which takes the initial value $f(\xi,x,0)$. From $f(\xi,x,0)$, we extract its moments $\rho(x,0)$, $u(x,0)$, $T(x,0)$ and solve the inviscid Euler equations with these initial values. Let $f_E(\xi,x,t)$ be the local Maxwellian formed from these Euler solutions $\rho(x,t)$, $u(x,t)$, $T(x,t)$. Then

$$\lim_{\varepsilon \to 0} f(\xi,x,t;\ \varepsilon) = f_E(\xi,x,t) \tag{6.1}$$

for any fixed $t > 0$ and in a variety of norms in $(x,\xi)$ (including $L_2$, maximum, and others). In the limit $\varepsilon \longrightarrow 0$ the Boltzmann equation disappears; but its solutions have a well defined limit. The moments $\rho$, $T$, $u$ are continuous at

$t = 0$ uniformly in $\varepsilon$, but $f$ is not; nor are $\partial u/\partial t$ and $\partial T/\partial t$.

The Navier-Stokes equations must be supplied with corrected initial data, $u_N(x,0;\ \varepsilon) = u_0(x,0) + \varepsilon u_1(x,0)$, etc. Here $u_0(x,0)$ is the actual initial velocity [a moment of $f(\xi,x,0)$], and the correction $u_1(x,0)$ is obtained from $f(\xi,x,0)$ by a definite analytic procedure. The solution $\rho_N(x,t;\ \varepsilon)$, $T_N(x,t;\ \varepsilon)$, $u_N(x,t;\ \varepsilon)$ of the Navier-Stokes equations (in which $\varepsilon$ multiplies the viscosity and heat conductivity) is used to construct a Chapman-Enskog distribution function $f_N = f_0 + \varepsilon f_1$ where $f_0$ is locally Maxwellian with $\rho_N$, $T_N$, $u_N$ as parameters, and $f_1$ depends on gradients of $u_N$ and $T_N$. It is then shown (in various norms) that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} |f - f_N| = 0 , \tag{6.2}$$

this time uniformly in $t > t_0$.

The quantities $\rho_1$, $T_1$, $u_1$, represent the initial slip, i.e., the discrepancy between the initial fluid state and the initial value which is required for the Navier-Stokes solution to correctly approximate the Boltzmann solution. The first order initial slip vanishes if $f(\xi,x,0)$ is a local Maxwellian. The initial slip vanishes to higher order if the initial distribution is a Chapman-Enskog

function; but this requires the initial function $f(\xi,x,0;\ \varepsilon)$ to depend explicitly on $\varepsilon$.

Proofs of these statements make use of strong existence theorems with estimates of f and of space derivatives of f that are uniform in the singular parameter $\varepsilon$.

The statement that a Boltzmann solution is approximated by fluid equations outside the regions of nonuniformity is somewhat flexible. The Hilbert theory (linear or nonlinear) gives a formal expansion of f in powers of $\varepsilon$ governed by the inviscid Euler equations to lowest order and by successive inhomogeneous Euler equations to higher order. More precisely, the higher order equations are the inhomogeneous variational equations of the Euler system (linear with variable coefficients, having the same characteristics as the Euler equations). The Chapman-Enskog theory also gives the Euler equations to lowest order; but this is followed by the Navier-Stokes equations, the Burnett equations, and a sequence of partial differential equations in $\rho$, T, and u of successively higher order in space derivatives. In other words, the Chapman-Enskog theory gives a formal expansion of the macroscopic equations which govern the fluid's evolution, whereas the Hilbert theory gives a formal expansion of the solution.

Both expansions, Hilbert and Chapman-Enskog, are asymptotic. But the ranges of validity are not the same. Each expansion, when truncated, satisfies the Boltzmann equation with a remainder,

$$\frac{\partial f_n}{\partial t} + \xi \cdot \frac{\partial f_n}{\partial x} + \frac{1}{\varepsilon} L(f_n) = \varepsilon^n R_n(\xi, x, t; \varepsilon) \tag{6.3}$$

The Hilbert remainder, which is obtained by solving inhomogeneous Euler equations, grows algebraically as $t^n$. The Navier-Stokes remainder decays as $t \longrightarrow \infty$. This is why the Hilbert expansion is asymptotic for only limited time, $t_0 < t < t_1$, whereas the Navier-Stokes result is uniformly valid for $t > t_0$. The higher order Chapman-Enskog approximations, Burnett, etc., are only formal at the present time in the absence of any estimates of the boundedness, growth, or decay of their solutions.

The proof that a given formal expansion is asymptotic to a solution of the Boltzmann equation is quite oblivious to the recipe which gave rise to the expansion. All that needs to be verified is that the expansion satisfies the Boltzmann equation term by term with a remainder $\varepsilon^n R_n(\xi, x, t; \varepsilon)$ where R is suitably bounded. We may construct such a series by ad hoc means as well as by expansion. For example, from the Hilbert or the Chapman-Enskog

formulas we can easily construct a unique sequence starting with Euler, Navier-Stokes, and continuing with inhomogeneous Navier-Stokes equations. It is trivial to verify that this sequence is asymptotically equivalent to either the Hilbert or Chapman-Enskog series. This formal series satisfies the Boltzmann equation with an error term $R_n$ which decays for large t at any order $n > 1$. This series is therefore superior to the Hilbert series in range of validity, and it is superior to the Chapman-Enskog series since the theory of the associated partial differential equations is known. Although this procedure is arbitrary for the initial value problem, we shall find that it arises naturally in slow steady flow (Sec. 8).

The solution of the initial slip problem can be given a simple abstract formulation in the case of the linear Boltzmann equation (5.4). In terms of a suitable normalization, the local Maxwellian takes the form

$$\omega = \frac{1}{(2\pi)^{3/2}} \exp(-\frac{1}{2}\xi^2) \tag{6.4}$$

and the nullspace of the collision operator L,

$$L(\psi_\alpha) = 0 , \qquad \alpha = 0,1,2,3,4 \tag{6.5}$$

is given by

$$\psi_0 = \omega^{1/2}$$

$$\psi_i = \xi_i \omega^{1/2} \qquad i = 1,2,3 \tag{6.6}$$

$$\psi_4 = (\xi^2 - 3)(\omega/6)^{1/2}$$

The $\psi_\alpha$ are orthonormal in the $L_2$ norm,

$$(f,g) = \int f\ g\ d\xi \tag{6.7}$$

The (linearized) fluid state is given by the projection of f into the nullspace of L,

$$\hat{f} = \sum \rho_\alpha \psi_\alpha \tag{6.8}$$

where

$$(\psi_0, f) = \rho_0 = \rho$$

$$(\psi_i, f) = \rho_i = u_i \qquad i = 1,2,3 \tag{6.9}$$

$$(\psi_4, f) = \rho_4 = (3/2)^{1/2} T$$

We introduce

$$f = \hat{f} + \overline{f} \tag{6.10}$$

where $\overline{f}$ is the projection into the orthogonal complement to the nullspace.

The basis of the Hilbert theory is the hypothesis that $f(\xi,x,t;\ \varepsilon)$ be expressible as a formal power series in $\varepsilon$. The essential conclusion is that to every given fluid state, $\rho_\alpha(x)$, is assigned a unique distribution function [Hilbert, 1912]

$$f_H(\xi,x;\ \varepsilon) = f_H^0 + \varepsilon f_H^1 + \ldots \tag{6.11}$$

An equivalent statement is that to a given projection $\hat{f}(\xi,x)$, there is assigned a unique distribution $f_H$; in particular, the leading term $f_H^0$ is exactly the given fluid state $\hat{f}$, and all succeeding terms are in the orthogonal complement. The manifold of special functions $f_H$ is formally invariant under solution of the Boltzmann equation. In other words, a special function $f_H$ taken as an initial value for the Boltzmann equation will give rise to a solution which remains within the space of functions $f_H$. A complementary result has also been proved [Grad, 1963a]. The hypothesis that $f(\xi,x,\tau;\ \varepsilon)$, (where $\tau = t/\varepsilon$) is a power series in $\varepsilon$ and that $f$ decays to zero implies that to a given projection $\overline{f}(\xi,x)$ is assigned a unique distribution function

$$f_G = f_G^0 + \varepsilon f_G^1 + \ldots \tag{6.12}$$

where $f_G^0$ is just the given projection $\overline{f}$, and all succeeding terms lie in the nullspace (i.e., they are strictly fluid components). This procedure also defines an asymptotically invariant space of solutions of the Boltzmann equation.

These two results can be combined to obtain the unique resolution

$$f(\xi,x) = f_H(\xi,x;\ \varepsilon) + f_G(\xi,x;\ \varepsilon) \tag{6.13}$$

of an arbitrary distribution f, which we choose to be independent of $\varepsilon$. To simplify the notation we write h for $f_H$ and g for $f_G$. The orthogonal decomposition $f = \overline{f} + \hat{f}$ of the given function $f(\xi,x)$ is used to obtain the leading terms

$$h_0^0 = \hat{f}\ , \quad g_0^0 = \overline{f} \tag{6.14}$$

which generate the two series

$$\begin{aligned} h_0(\xi,x;\ \varepsilon) &= h_0^0 + \varepsilon h_0^1 + \ldots \\ g_0(\xi,x;\ \varepsilon) &= g_0^0 + \varepsilon g_0^1 + \ldots \end{aligned} \tag{6.15}$$

The difference

$$f - h_0 - g_0 = \varepsilon f_1(\xi, x; \varepsilon) \tag{6.16}$$

is similarly decomposed, $f_1 = \bar{f}_1 + \hat{f}_1$ to generate the two series $h_1$ and $g_1$, etc. In this way we obtain (6.13) as the series

$$\begin{aligned} f_H &= f_0 + \varepsilon h_1 + \cdots \\ f_G &= g_0 + \varepsilon g_1 + \cdots \end{aligned} \tag{6.17}$$

Let $f(\xi,x,t; \varepsilon)$ be the solution of the Boltzmann equation with $f(\xi,x)$ as initial value. Taking as an initial value for the Boltzmann equation the Hilbert component $f_H$ of the resolution (6.13) of $f(\xi,x)$ yields a solution which is asymptotically equal to $f(\xi,x,t; \varepsilon)$ after a time $O(\varepsilon)$. The complementary solution, taking $f_G$ as initial value, describes a transient. Although the time evolution of $f_H$ is governed by fluid equations, the entire initial function $f(\xi,x)$ is reflected in the fluid initial data, $f_H(\xi,x; \varepsilon)$.

If the collision operator L has discrete eigenvalues other than zero, then the "transient" component $f_G$ can be further decomposed into fluidlike components which satisfy partial differential equations (similar to Euler,

Navier-Stokes, etc.), but which decay exponentially in time.

If $f(\xi,x)$ has n derivatives with respect to x, the separation into $f_H$ and $f_G$ can be carried out only to order $\varepsilon^{n-1}$. The problem of discontinuous or more general non-smooth initial data remains an open question.* For Maxwell molecules, the expansion $f_H$ is explicitly convergent, as a polynomial in $\varepsilon$, when $f(\xi,x)$ is a polynomial in x [Grad, 1963a]. One can expect convergence of the series for $f_H$ if f is sufficiently smooth (analyticity is not enough). A conjecture made in [Grad, 1963a], based on Maxwellian molecules, is that convergence requires f to be an entire function of x of order one-half. In the special case of hard spheres, a single Fourier coefficient, $f = \phi(\xi)e^{ikx}$, has been shown to give rise to a convergent series for $f_H$ when k is sufficiently small [McLennan, 1965]. It is not clear whether an entire function of order one (like $e^{ikx}$) is sufficient for convergence in general, or whether there is an essential difference between hard spheres and Maxwellian molecules.

*For a model of the Boltzmann equation, Y. Sone [Phys. Fluids, to appear] has obtained the asymptotic fluid behavior for a special class of discontinuous initial data.

## 7. The Boundary Value Problem--General Discussion

The boundary layer presents a more difficult problem than the initial layer for at least two reasons. First of all, the basic proof of existence is more difficult. There have been great strides recently in existence theory for the boundary value problem,* but these results are not yet adequate to the demands of the small mean free path limit. Neither the required uniform bounds as $\varepsilon \longrightarrow 0$ nor estimates of derivatives are yet available. This is not simply a technicality, since space derivatives do become infinite near a boundary.

The second difficulty is that the limit $\varepsilon \longrightarrow 0$ is ambiguous even in fluid dynamics. We recall that the viscosity and heat conductivity coefficients are proportional to $\varepsilon$. In a Stokes linearization about a free stream $U = 0$, the flow is essentially independent of value of the viscosity coefficient; cf. slow flow about a sphere or Poisseuille flow with a parabolic velocity profile. On the other hand, in an Oseen linearization about a finite

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*C. Cercignani (private communication) in a general bounded domain for the linear equation and Y. P. Pao (private communication) for the weakly nonlinear equation in a bounded one-dimensional domain.

velocity U, the limit $\mu \longrightarrow 0$ is singular and gives an inviscid limiting flow except within a boundary layer and wake. This boundary layer, of thickness $O(\varepsilon^{1/2})$, is distinct from the kinetic boundary layer which is $O(\varepsilon)$.

To obtain a definite result in the kinetic problem, we must at least distinguish between the various macroscopic cases. One distinction is the value of the Reynolds number. In the Stokes linearization we have set Re = 0 by setting the free stream velocity equal to zero before $\mu \longrightarrow 0$. In the Oseen linearization Re is finite, but it becomes infinite in the subsequent limit $\mu \longrightarrow 0$. We shall find it convenient to interpolate between these cases by taking an Oseen linearization after which we allow U to approach zero with $\varepsilon$. In this limit Re remains finite.

The interior of the fluid will be described by a formal expansion which is equivalent to Hilbert's in the initial value problem. By supplementing this expansion with conjectured existence theorems, we shall also be able to imitate the rigorous analysis of the initial layer by a plausible analysis of the boundary layer. This will give the asymptotic boundary conditions which are needed to complete the formal fluid expansion in the interior.

In the case of the Stokes linearization and the more general finite Re linearization just described, we find

that the formal Hilbert expansion automatically gives Navier-Stokes followed by successive inhomogeneous Navier-Stokes equations, rather than Euler and inhomogeneous Euler equations as in the initial value problem. The reason is that the steady flow Euler equations are too degenerate. They are not a determined system of differential equations which yield solutions merely by supplementation with boundary conditions. In common with most asymptotic expansions, a given term in the standard Hilbert expansion is not uniquely determined until we turn to the next higher order from which we extract compatibility conditions. In the case of steady flow, the additional degeneracy requires us to extract information from two succeeding levels to complete the determination of a given order. Thus the expansion of a solution of the Boltzmann equation in powers of $\varepsilon$ is essentially different in steady flow and time-dependent flow.

In the case $\mathrm{Re} \longrightarrow \infty$, a restriction to the viscous boundary layer rather than the entire interior allows a similar result (inhomogeneous Navier-Stokes equations) to be obtained [Trilling, 1964].

For the general case, nonlinear and nonsteady, we can find an asymptotic fluid description by an ad hoc procedure similar to the one described above (Sec. 6) in which

we proposed inhomogeneous Navier-Stokes equations for the initial value problem. The important point with regard to the boundary layer is that the boundary conditions which it must supply are, to any order in $\varepsilon$, just what is appropriate to a Navier-Stokes system.

## 8. Steady Hilbert Expansion for Finite Reynolds Number

We take the linear Boltzmann equation

$$\frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial x} + \frac{1}{\varepsilon} L(f) = 0 \tag{8.1}$$

in the normalized and dimensionless form described in equations (6.4) - (6.9). A Stokes linearization in steady flow is simply

$$\xi \cdot \frac{\partial f}{\partial x} + \frac{1}{\varepsilon} L(f) = 0 \tag{8.2}$$

and an Oseen linearization about the free stream velocity $\varepsilon U$ yields

$$\varepsilon U \cdot \frac{\partial f}{\partial x} + \xi \cdot \frac{\partial f}{\partial x} + \frac{1}{\varepsilon} L(f) = 0 \tag{8.3}$$

(the molecular velocity $\xi$ is relative to the stream at infinity, as is appropriate in taking moments of f). We note the linearized equation of state

$$p = \rho + T \tag{8.4}$$

and the conservation equations [appropriate to (8.1)]

$$\frac{\partial \rho}{\partial t} + \operatorname{div} u = 0$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial p}{\partial x_i} + \frac{\partial p_{ij}}{\partial x_j} = 0 \tag{8.5}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2} T\right) + \operatorname{div} u + \operatorname{div} q = 0$$

The dimensionless stress deviator and heat flow vector are the projections

$$p_{ij} = (\xi_i \xi_j - \frac{1}{3} \xi^2 \delta_{ij}, f)$$
$$q_i = (\frac{1}{2} [\xi^2 - 5] \xi_i, f) \tag{8.6}$$

The Euler equations are obtained by setting $p_{ij} = q_i = 0$ in (8.5), and the Navier-Stokes equations result from

$$p_{ij} = -\mu \left\{ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \operatorname{div} u \right\}$$
$$q_i = -\lambda \frac{\partial T}{\partial x_i} \tag{8.7}$$

The full Chapman-Enskog expansion takes the form [Grad, 1963a]

$$\frac{\partial \rho}{\partial t} + \operatorname{div} u = 0$$

$$\frac{\partial u}{\partial t} + \nabla p = \sum_{0}^{\infty} \varepsilon^{2n+1} \Delta^{n}[\mu_n \Delta u + \alpha_n \nabla(\operatorname{div} u)]$$

$$+ \sum_{1}^{\infty} \varepsilon^{2n} \Delta^{n}[\beta_n \nabla T + \gamma_n \nabla \rho] \qquad (8.8)$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2} T\right) + \operatorname{div} u = \sum_{0}^{\infty} \varepsilon^{2n+1} \Delta^{n+1} [\lambda_n T + \delta_n \rho]$$

$$+ \sum_{1}^{\infty} \varepsilon^{2n} \zeta_n \Delta^{n}(\operatorname{div} u)$$

where the coefficients $\mu_n$, $\lambda_n$, $\alpha_n$, $\beta_n$, $\gamma_n$, $\delta_n$, $\zeta_n$ depend on the molecular model. In particular, $\mu = \varepsilon\mu_0$ is the (dimensionless) viscosity coefficient and $\lambda = \varepsilon\lambda_0$ is the heat conductivity ($\alpha_0 = \mu_0/3$ and $\delta_0 = 0$). The steady flow Stokes linearization is obtained by setting $\partial/\partial t = 0$, and the Oseen linearization is obtained by replacing $\partial/\partial t$ by $U \cdot \nabla$.

In a steady Stokes flow the Euler equations are

$$\operatorname{div} u = 0$$
$$\nabla p = 0 \qquad (8.9)$$

and the Navier-Stokes equations are

$$\begin{aligned} &\operatorname{div} u = 0 \\ &\nabla p + \mu \operatorname{curl} \operatorname{curl} u = 0 \\ &\Delta T = 0 \end{aligned} \tag{8.10}$$

The conventional Hilbert expansion

$$f = \sum \varepsilon^n f_n \tag{8.11}$$

substituted into (8.2) yields the sequence of compatibility relations:

$$\begin{aligned} &\operatorname{div} u_0 = 0 \\ &\nabla p_0 = 0 \end{aligned} \tag{8.12}$$

at zero order and

$$\begin{aligned} &\operatorname{div} u_n = 0 \\ &\nabla p_n + \mu_0 \operatorname{curl} \operatorname{curl} u_{n-1} = 0 \\ &\Delta T_{n-1} = 0 \end{aligned} \tag{8.13}$$

for $n = 1, 2, \ldots$. We recall that $\mu_0 = \mu/\varepsilon$ is a scaled viscosity coefficient which is of order one.

Regrouping the variables,

$$\begin{aligned} &p_0 \\ &u_0,\ p_1,\ T_0 \\ &u_n,\ p_{n+1},\ T_n \qquad n = 1,2,\ldots \end{aligned} \tag{8.14}$$

gives $p_0$ = constant to lowest order and a well-posed Navier-Stokes system for each set $(u_n,\ p_{n+1},\ T_n)$.

The Stokes linearization gives a degenerate result in that the inhomogeneous terms vanish at any order* [this is easily seen by induction in (8.8)]. Since the same differential system occurs at each order, the sum of any finite number of terms in the Hilbert expansion (8.11) is exactly a Navier-Stokes flow; $\varepsilon$ enters only through the boundary conditions which must be found outside this theory and which we expect to be a formal power series in $\varepsilon$.

We turn to the Oseen linearization for which typical problems are the flow around a stationary object or the flow through a channel. Taking $\varepsilon U$ as the unperturbed

*This degeneracy was first pointed out in the special case of one-dimensional heat flow [Wang Chang and Uhlenbeck, 1953].

flow velocity, the formal Hilbert expansion (8.11) with the same grouping of variables (8.14) gives

$$p_0 = \text{constant} \tag{8.15}$$

followed by

$$\begin{aligned} \operatorname{div} u_0 &= 0 \\ U\cdot\nabla u_0 + \nabla p_1 + \mu_0 \operatorname{curl}\operatorname{curl} u_0 &= 0 \\ \tfrac{5}{2} U\cdot\nabla T_0 - \lambda_0 \Delta T_0 &= 0 \end{aligned} \tag{8.16}$$

then

$$\begin{aligned} \operatorname{div} u_1 &= U\cdot\nabla T_0 \\ U\cdot\nabla u_1 + \nabla p_2 + \mu_0 \operatorname{curl}\operatorname{curl} u_1 &= \sigma(U\cdot\nabla)\nabla T_0 \\ \tfrac{5}{2} U\cdot\nabla T_1 - \lambda_0 \Delta T_1 &= U\cdot\nabla p_1 \ , \end{aligned} \tag{8.17}$$

$[\sigma = \frac{1}{3}\mu_o + \frac{5}{2}(\beta_1 - \gamma_1)/\lambda_o]$ and further inhomogeneous forms of the same differential system. The system of equations (8.16) is the standard incompressible Oseen linearization of the Navier-Stokes equations (with scaled viscosity and heat conductivity).

In the general case, $Re \neq 0$, kinetic theory enters more substantially than in the special case $Re = 0$, where only the boundary conditions are modified. The reason,

qualitatively, is that the real characteristics $U \cdot \nabla$ of the Oseen linearization carry the boundary values into the flow.

## 9. Large Reynolds Number

With a finite value for the unperturbed speed U, the Reynolds number becomes large as $1/\varepsilon$, and there can be a double boundary layer. There is no advantage to linearization in this generality, so we may just as well consider the full nonlinear flow. A formal Hilbert expansion yields Euler equations and inhomogeneous Euler equations as in the initial value problem. This system is mathematically well posed provided that all streamlines intersect a boundary (or go to infinity). If we adopt the direct Hilbert expansion, we must look for asymptotic boundary conditions which are appropriate to the Euler equations, the boundary conditions to be extracted from an independent analysis of the combined viscous and kinetic boundary layers. A boundary condition on $u_n$ (reflecting the displaced boundary) will be a power series in $\varepsilon^{1/2}$. The first order term is taken from a classical viscous boundary layer analysis. The second order boundary term $O(\varepsilon)$ is a combination of a kinetic contribution and a second order viscous effect, and it is to be used in an inhomogeneous Euler equation. In particular, a second order fluid dynamic boundary layer analysis is not correct without the kinetic modification.

Another procedure is to give up the attempt to examine the whole flow and concentrate instead on the viscous boundary layer alone. The stretched variables in conventional

boundary layer analysis involve $\varepsilon$ (since $\mu \sim \varepsilon$). This reordering of terms, when combined with the Hilbert expansion, again gives Navier-Stokes (in a boundary layer approximation) and inhomogeneous Navier-Stokes equations [Trilling, 1964]. Any such higher order boundary layer analysis must be matched to an appropriate fluid formulation in the interior, e.g., inhomogeneous Euler or Navier-Stokes.

The most general procedure, applicable uniformly to the entire fluid domain, is to follow the ad hoc rule described previously. The complete (nonlinear and nonsteady) Chapman-Enskog expansion, involving differential equations of increasing order, is asymptotically equivalent to a sequence of Navier-Stokes and inhomogeneous Navier-Stokes equations with inhomogeneous terms of increasing order. Analysis of the kinetic boundary layer should give boundary conditions for the Navier-Stokes equations as a formal power series. The number of boundary conditions does not change with the order of approximation. Since $\varepsilon$ occurs nontrivially in the Navier-Stokes equations, their solutions will not be simple power series in $\varepsilon$. Nevertheless this procedure does produce a sequence of fluid approximations which is asymptotic to the solution of the Boltzmann equation to any order $\varepsilon^n$ in a region which includes the viscous boundary layer and excludes only the kinetic boundary layer.

It is important to note that there is an essentially arbitrary choice of fluid procedures available for approximating to the interior fluid flow, excluding the kinetic boundary layer. The number and type of boundary conditions that must be supplied by the kinetic boundary layer analysis depends on the particular fluid choice that is made (Euler, Navier-Stokes, Burnett, etc.). In other words, the kinetic boundary layer analysis cannot be made universally but only in conjunction with a particular treatment of the interior. The common worry that Burnett and higher order Chapman-Enskog equations presumably require an increasing number of boundary conditions poses no real difficulty. There is no compelling reason to choose the Chapman-Enskog procedure which is only one of a number of valid formal series. The choice we recommend, a sequence of Navier-Stokes systems with inhomogeneous terms of increasing order and similar slip boundary conditions, has the overwhelming advantage of including the viscous boundary layer (as distinguished from the Hilbert expansion) and of utilizing a familiar system of differential equations (as opposed to the Chapman-Enskog expansion). For reasons that will appear in the next section, the inhomogeneous terms in the differential equations and the finite slip boundary conditions enter nontrivially at the same order.

## 10. The Boundary Layer--Slip Flow

The initial layer is studied by introducing a stretched time scale, $\tau = t/\varepsilon$, in terms of which the Boltzmann equation

$$\frac{\partial f}{\partial \tau} + L(f) = -\varepsilon\, \xi \cdot \frac{\partial f}{\partial x} \tag{10.1}$$

exhibits weak coupling of spatial gradients with a purely time-dependent problem. Similarly, the boundary layer can be studied by introducing $y = x_1/\varepsilon$ as the scaled distance from the wall and $x' = (x_2, x_3)$ as unscaled coordinates within a parallel surface $x_1$ = constant to obtain an approximately one-dimensional Boltzmann equation

$$\xi_1 \frac{\partial f}{\partial y} + L(f) = O(\varepsilon) \tag{10.2}$$

This estimate [ignoring the term $O(\varepsilon)$] is sufficient to calculate first order slip coefficients. We can easily refine this estimate

$$\xi_1 \frac{\partial f}{\partial y} + L(f) = -\varepsilon \xi' \cdot \frac{\partial f}{\partial x'} + O(\varepsilon^2) \tag{10.3}$$

In order to simplify the calculations (in particular keeping all tensor moments of f unchanged), we do not

transform $\xi$, keeping the original rectangular coordinates in velocity space.

The slip boundary conditions arise from matching solutions of the boundary layer equation (10.3) with the Hilbert expansion in the interior. To do this we shall have to postulate certain existence theorems, both for the full Boltzmann equation in a general interior domain D, and for the one-dimensional equation in a half space. The two types of existence theorems are quite distinct. For example, $u_n = 0$ can be imposed on the entire boundary of D, but conservation of mass implies $u_n = 0$ at one end (or at infinity) if it holds at the opposite end of a one-dimensional problem. Also, suitable regularity and boundedness conditions at infinity are needed for the unbounded domain.

The basic boundary condition involves specifying the half distribution function $f^-$, $\xi \cdot n < 0$, where n is the outward normal to D. To be able to guarantee mass conservation ($u_n = 0$) at the boundary, we specify $f^-$ modulo a constant multiple of $\psi_0$ [cf. (6.6)]

$$f^- = \phi^- + c_0\psi_0 \tag{10.4}$$

The parameter $c_0$ (which is a point function over the boundary of the domain) is to be adjusted to satisfy

$u_n = 0$. A special case of this boundary condition is diffuse reflection for which $\phi^-$ is explicit,

$$f^- = c_0\psi_0 + u_w' \cdot \psi' + (3/2)^{1/2}\, T_w\psi_4 \qquad (10.5)$$

Here $u_w'$ and $T_w$ are the tangential wall velocity and the wall temperature, and $\psi'$ is the tangential component of the vector $\psi_i$ (6.6). The parameters $u_w'$ and $T_w$ are given functions on the boundary, and $c_0$ is to be determined such that $u_n = 0$. This is the linearized version of the statement that the emitted distribution is a Maxwellian at the local wall velocity and temperature.

We now list the conjectured theorems for an interior domain D:

Theorem 1: A unique steady solution exists subject to $f^-$ given on the boundary of D.

Theorem 2: A steady solution exists with $u_n = 0$ subject to (10.4) where $\phi^-$ is given and $c_0$ is left open; the solution is unique if the total mass is specified, $\int_D \rho\, dx =$ given (or $\int_D p\, dx =$ given).

Corollary: A unique solution exists subject to the diffuse boundary condition (10.5) with specified wall velocity and temperature, provided that the total mass of the system is specified.

The result quoted in Theorem 1 is a proved theorem in a certain weighted $L_2$ norm.* In the same norm, Theorem 2 and its corollary can probably be proved using the methods of [Grad, 1966].

In a one-dimensional steady flow there are five constants of the motion. In a nonlinear version these are

$$\begin{gathered} \rho u_x \\ \rho u_x^2 + p + p_{xx} \\ \rho u_x u_y + p_{xy} \\ \rho u_x u_z + p_{xz} \\ \rho u_x (e + \tfrac{1}{2} u^2) + u_x (p + p_{xx}) + q_x \end{gathered} \tag{10.6}$$

and in the linearized version (about zero free stream) they can be taken as

$$\begin{gathered} u_x \\ p + p_{xx} \\ p_{xy} \ , \ \ p_{xz} \\ q_x \end{gathered} \tag{10.7}$$

*C. Cercignani, private communication.

The basic conjectured existence theorem is

Theorem 3: A unique steady solution exists in a half space $x > 0$ if at $x = 0$, $f^-$ is given modulo $c_0\psi_0$ (10.4), and if, in addition, the five constants (10.7) are specified.

Far from the imposed boundary condition at $x = 0$, we expect a Navier-Stokes flow with linear temperature gradient, $dT/dx = \text{const.}$, and velocity gradient, $du'/dx = \text{const.}$, $u' = (u_y, u_z)$. Setting three of the constants equal to zero,

$$p_{xy} = p_{xz} = q_x = 0 \qquad (10.8)$$

guarantees that the flow is bounded at infinity. The flow at infinity will then be a constant Maxwellian characterized by five parameters. To produce a flow which decays to zero at infinity, we must impose five additional conditions. There are two constants (10.7) left which we can set equal to zero, $u_x = p = 0$. This leaves three finite constants, $u'$ and $T$, at infinity. Subtracting these from the given boundary function $f^-$, we are led to

Theorem 4: There exists a unique steady solution in the half space, vanishing at infinity and satisfying the boundary condition

$$f^- = \phi^- + c_0\psi_0 - u'\cdot\psi' - (3/2)^{1/2}\, T\psi_4 \qquad (10.9)$$

where $\phi^-$ is arbitrarily given and $(c_0, u', T)$ are adjustable parameters.

The interpretation of Theorem 4 is that to any given emitted distribution $\phi^-$ are assigned unique values of the slip velocity $u'$ and temperature T. The parameter count (which would give <u>four</u> boundary conditions at infinity for the elementary boundary condition, $f^- =$ given) can be easily discovered as a plausible consequence of the basic Theorem 1 applied to a finite one-dimensional domain, say $0 < x < x_1$, on letting $x_1 \to \infty$.* The application of Theorem 2 to this domain is somewhat different. Two adjustable constants $c_0$ are available at the two ends even though $u_x = 0$ at one end implies $u_x = 0$ at the other. The counting is correct, however, since the second constant $c_0$ is needed to adjust the total mass, $\int \rho \, dx$.

We shall also refer to the

Corollary: Theorem 4 holds if the Boltzmann equation is replaced by an inhomogeneous equation

$$\xi_1 \frac{\partial f}{\partial y} + L(f) = \psi \qquad (10.10)$$

where $\psi$ decays rapidly as $y \to \infty$. In this case the moments (10.7) are not constant but are taken to be zero at $y = \infty$.

*In a special case in which the parameter count is simpler, this theorem has been proved for a model of the Boltzmann equation; Y. P. Pao, private communication.

The slip coefficients, $u'$ and $T$ in (10.9), will depend linearly on the given boundary function $\phi^-$ and the inhomogeneous term $\psi$.

Let us now consider a general domain D with smooth boundary, taken with the diffuse reflection boundary condition (10.5), $u_w'$ and $T_w$ being specified on the boundary; (the boundary may be interior or exterior and may have disjoint components, as, for example, in a flow between two cylinders or spheres). The Boltzmann equation is to be solved in the original scaling,

$$\xi \cdot \frac{\partial f}{\partial x} + \frac{1}{\varepsilon} L(f) = 0 \ . \qquad (10.11)$$

As in the initial layer analysis, we iterate back and forth between the interior and boundary layer. To start, we solve the Navier-Stokes equations in the given domain D subject to $u_n = 0$ and the given boundary conditions for $u_w'$ and $T_w$. Using this solution we construct a Chapman-Enskog function $f_0 + \varepsilon f_1$; $f_0$ is a local Maxwellian with the Navier-Stokes parameters, and $f_1$ is a more complicated function of $\xi$ (obtained by solving integral equations) involving gradients of the Navier-Stokes velocity and temperature. The Maxwellian function $f_0$ satisfies the kinetic boundary condition (10.5) exactly (and determines a specific value for the parameter $c_0$). The difference between the exact solution

to the Boltzmann equation and the Navier-Stokes approximation,

$$f - (f_0 + \varepsilon f_1) = \varepsilon g \ , \qquad (10.12)$$

satisfies an inhomogeneous Boltzmann equation (related to the Burnett approximation) and is subject to a boundary condition induced by the Navier-Stokes residual $f_1$,

$$g^- = c_1 \psi_0 - f_1^- \qquad (10.13)$$

Turning to the one-dimensional boundary layer equation (which to this order is simply the homogeneous one-dimensional Boltzmann equation), we find a solution $g_0$ (Theorem 4) which decays at infinity and satisfies the boundary condition

$$g_0^- = -f_1^- + c_1 \psi_0 - u_1' \cdot \psi' - (3/2)^{1/2} T_1 \psi_4 \qquad (10.14)$$

Having determined the slip coefficients $u_1'$ and $T_1$ (also $c_1$), we then compute a Navier-Stokes function $g_1$ in D which takes the boundary values $u_1'$, $T_1$ (and $u_n = 0$). The sum, $g_0 + g_1 = g$ satisfies the boundary condition (10.13) (including the determination of $c_1$) and leaves a residue which is $O(\varepsilon^2)$.

The first-order slip coefficients $u_1'$ and $T_1$ are computed as the solution of a one-dimensional problem (Theorem 4) with a boundary condition $f_1^-$ which is taken from the first-order Chapman-Enskog function. This function, $f_1$, is a linear combination of velocity and temperature gradients, $\partial u_i/\partial x_j$ and $\partial T/\partial x_i$ (or $p_{ij}$ and $q_i$), taken from the zero-order Navier-Stokes solution. A scalar $T_1$ and tangential vector $u_1'$ can be combined with the normal $n_i$ as linear functions of $p_{ij}$ and $q_i$ only in the combinations

$$\begin{aligned} T_1 &= \alpha\tau_n + \beta q_n \\ u_1' &= \gamma\tau' + \delta q' \end{aligned} \tag{10.15}$$

where $\tau_i = p_{ij}n_j$ is the vector force on the wall, and $\tau_n = \tau\cdot n$, $\tau' = \tau - n\tau_n$ are its normal and tangential components. The coefficient $\beta$ is the conventional temperature slip due to normal heat flow, and $\gamma$ describes the slip velocity $u_1'$ induced by the tangential stress $\tau'$. The slip $u_1'$ caused by $q'$ is called thermal creep.

The term $\alpha\tau_n$ is ostensibly of higher order since $\tau_n$ always vanishes at the boundary in any Navier-Stokes steady flow with no-slip boundary conditions.* But it would be

*This was pointed out to the author by Y. Sone.

retained in a time-dependent boundary value problem, or in any problem (such as with the thirteen moment equations) in which no strict ordering with respect to a parameter is followed.

The formulation for second and higher order slip boundary conditions is quite similar. As in the analysis of the initial layer, we note that both the interior and boundary layer expansions individually satisfy the Boltzmann equation without interference. The two expansions are coupled only through boundary conditions. The interior (Hilbert) expansion can be written

$$\begin{aligned} f_H = f_0^0 + \varepsilon f_1^0 + \varepsilon^2 f_2^0 + \ldots \\ + \varepsilon f_0^1 + \varepsilon^2 f_1^1 + \ldots \\ + \varepsilon^2 f_0^2 + \ldots \end{aligned} \qquad (10.16)$$

where $f_0^0$, $f_0^1$, $f_0^2$ are locally Maxwellian; $f_1^0$ and $f_2^0$ are the Navier-Stokes and Burnett residues induced by $f_0^0$; $f_1^1$ is the Navier-Stokes residue induced by $f_0^1$. All the terms on a given line in (10.16) are uniquely determined once the leading (Maxwellian) term is specified. The Maxwellian $f_0^0$ is a solution of the Navier-Stokes equations subject to the given boundary conditions, $u_w'$ and $T_w$. The higher order Maxwellians $f_0^1$ and $f_0^2$ satisfy the Navier-Stokes equations,

but with slip boundary conditions $u_1'$, $T_1$ and $u_2'$, $T_2$ which are to be found from the boundary layer analysis.

The determination of $f_0^1$ (from the first order boundary conditions, $u_1'$ and $T_1$) is described above. The second order boundary conditions for $f_0^2$ are found (Theorem 4, Corollary) by solving the inhomogeneous one-dimensional boundary layer equation

$$\xi_1 \frac{\partial g^1}{\partial y} + L(g^1) = -\xi' \cdot \frac{\partial g^0}{\partial x'} \qquad (10.17)$$

where $g^0$ is the already known first order boundary layer solution [called $g_0$ in (10.14)], and $g^1$ is to be found subject to the boundary condition

$$(g^1)^- = -(f_2^0 + f_1^1)^- + c\psi_0 - u_2' \cdot \psi' - (3/2)^{1/2} T_2 \psi_4 \qquad (10.18)$$

The Navier-Stokes term $(f_1^1)^-$ in the boundary condition leads to exactly the same slip coefficients $(\alpha,\beta,\gamma,\delta)$ as in (10.15). The Burnett function $f_2^0$ involves second derivatives of $u^0$ and $T^0$ (or first derivatives of $q_i^0$ and $p_{ij}^0$) and gives rise to seven slip coefficients:

$$T_2 = ak_n + b\sigma_n + c(n \cdot \nabla p)$$

$$u_2' = dk' + e\sigma' + f\nabla' p + g\theta' \qquad (10.19)$$

where

$$\sigma_i = n_r n_s \frac{\partial p_{ir}}{\partial x_s}$$

$$\theta_i = n_r n_s \frac{\partial p_{rs}}{\partial x_i} \quad (\sigma_n = \theta_n) \tag{10.20}$$

$$k_i = n_r \frac{\partial q_i}{\partial x_r}$$

The terms in $\nabla p$ are alternative forms equivalent to $\partial p_{ij}/\partial x_j$. The function $g^0$ (generated from $f_1^0$ as a boundary condition) takes the form

$$\begin{aligned} g^0 = {} & p_{ij}(x_2,x_3)\xi_i\xi_j \, \phi(\xi_1, \xi_2^2 + \xi_3^2, y) \\ & + q_i(x_2,x_3)\xi_i \, \psi(\xi_1, \xi_2^2 + \xi_3^2, y) \end{aligned} \tag{10.21}$$

In evaluating $\xi' \cdot \partial g^0/\partial x'$ for (10.17) we have contributions from $\xi' \cdot \partial p_{ij}/\partial x'$ and $\xi' \cdot \partial q_i/\partial x'$ which are easily seen to be included within the format (10.19) (in fact, only five of the possible seven slip coefficients arise from this source, since $\sigma$ and $\nabla p$ occur only through the combination $\sigma + \nabla p$). But there are also contributions from $\xi' \cdot \partial\phi/\partial x'$ and $\xi' \cdot \partial\psi/\partial x'$ which are derivatives holding $\xi$ fixed in a given rectangular coordinate system, whereas the components $\xi_1 = \xi \cdot n$ and $\xi_2^2 + \xi_3^2 = |\xi \times n|^2$ vary with the normal n to the boundary. A simple computation gives the result

$$\xi' \cdot \frac{\partial \phi}{\partial x'} = (\phi_1 - \xi_n \phi_2)(\xi_i \kappa_{ij} \xi_j) \qquad (10.22)$$

and similarly for $\psi$, where $\phi_1$ and $\phi_2$ are the derivatives of $\phi$ with respect to its arguments $\xi_1$ and $\xi_2^2 + \xi_3^2$ respectively and

$$\kappa_{ij} = \kappa' a_i a_j + \kappa'' b_i b_j \qquad (10.23)$$

describes the curvature of the boundary of the domain; $\kappa'$ and $\kappa''$ are the principal curvatures along the principal directions $a_i$ and $b_i$ and $\kappa_{ij} t_j$ is the geodesic curvature in a tangent direction $t_j$; also $\kappa'$ and $\kappa''$ are positive if the geodesic curvature is toward the interior of the domain. From the products $p_{ij}\kappa_{rs}$ and $q_i \kappa_{rs}$ in $\xi' \cdot \partial g^0/\partial x'$ we obtain seven further slip coefficients,

$$T_2 = \alpha_2 \tau_n \bar{\kappa} + \beta_2 q_n \bar{\kappa} + \lambda_2 p_{rs} \kappa_{rs}$$
$$u_2' = \gamma_2 \tau' \bar{\kappa} + \delta_2 q' \bar{\kappa} + \mu_2 \kappa \cdot \tau + \nu_2 \, \kappa \cdot q \qquad (10.24)$$

where

$$\bar{\kappa} = \frac{1}{2} \kappa_{rr} = \frac{1}{2}(\kappa' + \kappa'') \qquad (10.25)$$

is the mean curvature (note that $\kappa \cdot \tau = \kappa_{ij}\tau_j$ and $\kappa \cdot q$ are necessarily tangential to the boundary.) The four coefficients $\alpha_2$, $\beta_2$, $\gamma_2$, $\delta_2$ are similar in effect to $\alpha$, $\beta$, $\gamma$, $\delta$,

but with an extra factor $\bar{\kappa}$.

Since the second order boundary layer equation (10.17) is inhomogeneous, the moments (10.7) which are zero at $y = \infty$ are finite at $y = 0$ (cf. Corollary to Theorem 4). The normal velocity, $u_n$, is therefore different from zero [the adjustable constant c in (10.18) is required to make $u_n = 0$ at $y = \infty$]. We have an effective displacement of the boundary given by the Navier-Stokes slip condition

$$u_n = \hat{a}(k_n + 2\bar{\kappa}q_n) + \hat{b}(\sigma_n + 2\bar{\kappa}\tau_n + \frac{\partial p}{\partial x_n} - \kappa_{rs}p_{rs}) \qquad (10.26)$$

The presence of only two slip coefficients is a consequence of div u = 0 which allows $u_n$ to be evaluated in terms of the first order mean slip velocity $\hat{u}' = \int_0^\infty u'(y)dy$ [obtained from $\int_0^\infty g^0(y)dy$ in (10.21)] using $\oint \hat{u}' \cdot dx$ for paths on the boundary. In computing drag, heat transfer at the boundary, etc., the jump in the values of other moments (10.17) across the boundary layer would be needed.

The slip coefficients g in (10.19), $\alpha_2$ and $\lambda_2$ in (10.24), and the terms $2\bar{\kappa}\tau_n - \kappa_{rs}p_{rs}$ in (10.26) can be considered to be of higher order just as was pointed out for the first order coefficient $\alpha$.

In addition to the explicit appearance of boundary curvature in (10.24), it is also implicit in the Navier-Stokes expression for $p_{ij}$ in terms of $\partial u_i/\partial x_j$ and also in

the global relation between $T_w$ and the boundary values of $q_n$ and $k_n$ through the solution of an elliptic equation.

A rough estimate can be made of the first order slip coefficients following Maxwell; for diffuse reflection (cf. [Grad, 1958])

$$\alpha = -1/4 \qquad \beta = (\pi/8)^{1/2} \sim 0.627$$
$$\gamma = (\pi/2)^{1/2} \sim 1.235 \qquad \delta = -1/5 \tag{10.27}$$

More accurate values have been computed for $\beta$ and $\gamma$ using half range polynomial expansions; for Maxwellian molecules [Ziering, 1960]

$$\beta \sim 0.766$$
$$\gamma \sim 1.48 \tag{10.28}$$

For a relaxation model of the Boltzmann equation essentially precise values have been calculated for $\beta$ [Welander, 1954; Sone, 1966a], for $\gamma$ [Willis, 1960; Albertoni, et al., 1963], and for $\delta$ [Sone, 1966b],

$$\beta \sim 0.737$$
$$\gamma \sim 1.437 \qquad \delta \sim -0.306 \tag{10.29}$$

and for five of the sixteen second order coefficients; e-f [Cercignani and Sernagiotto, 1966], (2b-c) and $\hat{b}$ [Tamada, 1967], ($\gamma_2$ + 2e - 4f) and $\delta_2$ [Sone and Yamamato, private communication],

$$2b - c \sim -0.1980\lambda \qquad e-f \sim 0.960\lambda$$
$$\gamma_2 + 2e - 4f \sim 2.55\lambda \qquad \delta_2 \sim 0.189\lambda \qquad (10.30)$$
$$\hat{b} \sim 0.2927\lambda$$

The normalization (10.15) in terms of stress and heat flow instead of velocity and temperature gradients makes the values of the first order slip coefficients independent of the essentially arbitrary parameter which enters into a relaxation model. But the multiplicative factor for the second order coefficients depends on this parameter, i.e., on the definition of a mean free path [which in (10.30) is $\lambda = (8RT/\pi)^{1/2}/\nu$].

We have found by consistent formal expansion that the Navier-Stokes system with no-slip boundary conditions is followed, at next order, by an inhomogeneous Navier-Stokes system (comparable to Burnett) subject to the slip boundary conditions (10.15). In particular, slip boundary conditions are formally inconsistent with use of the ordinary Navier-Stokes equations. This is in contrast to the initial value problem where the first occurrence of the homogeneous Navier-Stokes equations is accompanied by nontrivial slip

initial conditions. The reason for the discrepancy is that the standard diffuse reflection boundary condition states that $f^-$ is a local Maxwellian. If the initial function $f(\xi,x,0)$ is taken to be locally Maxwellian, the initial slip also recedes one notch, and it appears for the first time at the Burnett (or inhomogeneous Navier-Stokes) level.

We have presented the boundary layer theory for a Stokes linearization of the fluid equations. It is applicable after trivial modifications [viz. restoring $\partial p_{ij}/\partial x_j$ for $\partial p/\partial x_i$ in (10.19)] for an Oseen linearization and, to first order, even for the nonlinear steady Navier-Stokes equations. In the nonlinear case, the expansion becomes one in $\varepsilon^{1/2}$ rather than $\varepsilon$. The linear first order slip theory is valid since the kinetic layer is thinner than the viscous layer by a factor $O(\varepsilon^{1/2})$ (we require that $M\varepsilon^{1/2}$ be small). The entire analysis is for smooth boundaries; the behavior near a corner, leading edge, etc., requires separate investigation. In a time-dependent problem, the first order slip boundary condition (10.15) is unchanged (after leaving the initial layer). The second order theory is modified by the occurrence of an additional inhomogeneous term, $-\partial g^0/\partial t$, in (10.17).

Implicit in the analysis made above is a conjectured existence theorem for the steady Navier-Stokes equations in an interior domain D given the vector velocity and temperature on the boundary and the total mass in D.

## 11. Infinitely Strong Shock Profile

We recall that for a weak shock the Navier-Stokes profile is explicit and universal [Taylor, 1910; Grad 1952],

$$\phi(x) = \tanh \varepsilon x/\lambda^* \tag{11.1}$$

Here $\phi$ represents any of the flow variables, density, pressure, temperature, or velocity, suitably normalized to vary between -1 and +1; $\varepsilon$ is a shock strength parameter, $0 < \varepsilon < 1$, e.g.

$$\varepsilon = \frac{p_1 - p_0}{p_1 + p_0} \tag{11.2}$$

(the subscripts 0 and 1 refer to the front-upstream and back-downstream respectively); $\lambda^*$ is a specific mean free path. Expressed in terms of the viscosity coefficient

$$\lambda^* = \frac{35}{12} a_0 \frac{\mu}{m} \tag{11.3}$$

where $m = \rho u$ is the mass flow constant and $a_0$ is a numerical constant which is exactly unity for Maxwellian molecules ($\mu \sim T$) and differs from unity by less than

five percent for all potentials between Maxwellian and hard spheres ($\mu \sim T^{1/2}$).

The weak shock profile can also be found from the Boltzmann equation by a formal expansion in $\varepsilon$.* Scaling $\varepsilon x = y$ yields a Boltzmann equation

$$\xi \frac{\partial f}{\partial y} = \frac{1}{\varepsilon} Q(f,f) \qquad (11.4)$$

in which the parameter $\varepsilon$ appears exactly as in the Hilbert theory. But, as in the steady flow linearization (Sec. 8), the fluid compatibility relations are degenerate, and we do not obtain the standard Hilbert or Chapman-Enskog sequence of fluid equations. The steady flow Euler equations in one dimension can be completely integrated. The compatibility relations are therefore essentially algebraic rather than differential equations. The lowest order compatibility requirement (corresponding to a local Maxwellian $f_o$) yields a constant state. To next order, the variational Euler equations are a linear homogeneous algebraic system; the condition for a nontrivial solution (vanishing determinant) is that the zero order constant state be sonic. We recognize this as a bifurcation equation;

*P.N. Hu, private communication

in searching for a weak shock, we are looking for a non-unique solution which branches off from the trivial constant state. To next order the classical G.I. Taylor solution (11.1) is found. This expansion in $\varepsilon$ is not strictly comparable to Navier-Stokes, Burnett, etc. since the compatibility relations are not a sequence of fluid dynamic equations. But a similar singular perturbation applied to the Navier-Stokes equations will yield the same leading term; applied to the thirteen moment equations it yields two correct terms, etc.*

For finite shock strength the Navier-Stokes profile can be solved numerically by integration of a simple plane direction field [Gilbarg and Paolucci, 1953] or by expansion in $\varepsilon$ [Grad, 1952] or even explicitly for the special value $a_0 = 20/21$ [Becker, 1922; Thomas, 1944]. A paradoxical early result was that the shock thickness for strong shocks could become arbitrarily small or arbitrarily large compared to the mean free path, depending on the temperature dependence of the viscosity coefficient. This difficulty was resolved [Grad, 1952] by the simple expedient of introducing the downstream mean free path, $\lambda_1$, as the reference length

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*P.N. Hu, private communication

instead of the upstream mean free path $\lambda_0$. Not only is the limiting thickness finite in terms of $\lambda_1$, but there is a definite limiting profile for each of the flow variables. The importance of this choice of scaling lies in the fact that $\lambda_1/\lambda_0$ is unbounded in a strong shock (except in the special case of hard spheres).

We now suggest that for the Boltzmann equation there is a finite limiting profile at infinite strength for the distribution function $f(\xi,x)$ when the correct scaling is introduced in velocity space, viz. at the downstream sound-speed, $c_1 = (5/3\ RT_1)^{1/2}$. With this scaling, the upstream Maxwellian reduces to a $\delta$-function ($c_0 \sim 0$) centered at $u_0/c_1 = 4/\sqrt{5}$ ; cf. Fig. 1 where the upstream Maxwellian is indicated by constant f contours centered at $u_1/c_1 = 1/\sqrt{5}$ . The crucial point is that the entire upstream distribution is at a definite thermal velocity when measured on the downstream scale. Even though the mean free path is strongly velocity dependent, the mean free path of an upstream particle when considered to impinge upon the downstream distribution is, within a factor of order unity, equal to the mean free path for self relaxation of the upstream distribution.

Based on this qualitative remark, we introduce the Ansatz

$$\bar{f} = f_\delta + f \tag{11.5}$$

where

$$f_\delta = \rho_\delta(x)\delta(\xi - u_0) \tag{11.6}$$

and f is taken to be nonsingular on the downstream scale (it may be unbounded near $\xi = u_0$, but it is, nevertheless, an integrable function and not a distribution).

For any cutoff potential we write

$$Q(\bar{f},\bar{f}) = Q_1(\bar{f},\bar{f}) - \bar{\nu}\bar{f} \tag{11.7}$$

where

$$Q_1(\bar{f},\bar{f}) = \int \bar{f}'\bar{f}' \; V \, d\omega d\xi_1 \tag{11.8}$$

$$\bar{\nu}(\xi,x) = \int \bar{f}_1 \; V \, d\omega d\xi_1 = \nu + \nu_\delta \tag{11.9}$$

in a standard notation. In the expansion of the collision term $Q(f_\delta + f, f_\delta + f)$ we indentify $Q(f_\delta, f_\delta)$ and $\nu f_\delta$ as $\delta$-functions (a collision between two molecules in $f_\delta$ remains within $f_\delta$) and all other terms as legitimate

functions on the scale f. Consequently, the Boltzmann equation splits into

$$u_0 \frac{\partial f_\delta}{\partial x} + \nu_0 f_\delta = Q(f_\delta, f_\delta) \tag{11.10}$$

and

$$\xi \frac{\partial f}{\partial x} + \nu_\delta f - 2Q_1(f_\delta, f) = Q(f,f) \tag{11.11}$$

where, in $\nu(\xi,x)$ we set $\xi = u_0$,

$$\nu_0(x) = \nu(u_0,x) \tag{11.12}$$

The mass conservation moment of (11.10) is

$$u_0 \frac{d\rho_\delta}{dx} + \nu_0 \rho_\delta = 0 \tag{11.13}$$

We have

$$\rho_\delta(x) = \rho_0 \exp\left(-\frac{1}{u_0} \int_{-\infty}^{x} \nu_0 dx\right) \tag{11.14}$$

and therefore $f_\delta$ (11.6) is given explicitly in terms of the "moment" $\nu_0$ of f. Substituting this expression for $f_\delta$

into (11.11), we obtain a nonlinear integro-differential equation in f alone. This Boltzmann-like equation, provided that it has a solution satisfying the boundary conditions

$$f(-\infty) = 0$$

$$f(+\infty) = f_1 = \frac{\rho_1}{(2\pi RT_1)^{3/2}} \exp[-\frac{(\xi-u_1)^2}{2RT_1}] \tag{11.15}$$

gives the solution to the infinite strength shock profile. No shock strength parameter remains, and the downstream parameters $\rho_1$, $u_1$, $RT_1$, are easily scaled out of the problem. The only parameters that remain are those implicit in the molecular force law. The equation (11.11) for f is, in appearance, at least as complicated as the original Boltzmann equation. But it has the great advantage of being nonsingular, or, at least, less singular than the Boltzmann equation. In particular, this will allow us to use naive approximations for f which would be worthless for the evaluation of $\overline{f}$.

The simplest approximation to any distribution f is the locally Maxwellian; in one dimension this involves three moments. For the total distribution, $\overline{f}$, this is too crude to give a profile at any strength; it leads at

most to the jump conditions and a discontinuous shock front. Applied to f, this approximation becomes valuable.

Taking the three moments $(1, \xi_1, \frac{1}{2}\xi^2)$ of the full Boltzmann equation, without approximation, yields conservation of mass, momentum, and energy,

$$\begin{aligned} \int \xi_1 \, \overline{f} \, d\xi &= m \\ \int \xi_1^2 \, \overline{f} \, d\xi &= P \\ \int \tfrac{1}{2} \xi_1 \xi^2 \, \overline{f} \, d\xi &= Q \end{aligned} \tag{11.16}$$

where m, P, Q are constants. For $f_\delta$ these moments (x-dependent) are

$$\begin{aligned} \int \xi_1 \, f_\delta d\xi &= \rho_\delta u_0 \\ \int \xi_1^2 \, f_\delta d\xi &= \rho_\delta u_0^2 \\ \int \tfrac{1}{2} \xi_1 \xi^2 \, f_\delta d\xi &= \tfrac{1}{2} \rho_\delta u_0^3 \end{aligned} \tag{11.17}$$

For f, using the locally Maxwellian approximation, we have

$$\int \xi_1 \, f d\xi = \rho u$$

$$\int \xi_1^2 \, f d\xi = \rho u^2 + p \qquad (11.18)$$

$$\int \frac{1}{2} \xi_1 \xi^2 d\xi = \frac{1}{2} \rho u^3 + \frac{5}{2} pu$$

where $\rho(x)$, $u(x)$, $p(x)$ are the standard moments of $f$. The constants $m$, $P$, $Q$ can be evaluated at $x = -\infty$ where $\overline{f} = f_\delta$,

$$m = \rho_0 u_0$$

$$P = \rho_0 u_0^2 \qquad (11.19)$$

$$Q = \frac{1}{2} \rho_0 u_0^3$$

Adding (11.17) to (11.18), we obtain

$$\rho_\delta u_0 + \rho u = \rho_0 u_0$$

$$\rho_\delta u_0^2 + \rho u^2 + p = \rho_0 u_0^2 \qquad (11.20)$$

$$\rho_\delta u_0^3 + \rho u^3 + 5pu = \rho_0 u_0^3$$

Eliminating $p$ and $\rho$ from these equations leaves an equation in $u$,

$$(\rho_\delta - \rho_0)u_0(4u^2 - 5uu_0 + u_0^2) = 0 \tag{11.21}$$

If $\rho_\delta$ is not identically equal to $\rho_0$, then

$$4u^2 - 5uu_0 + u_0^2 = 0 \tag{11.22}$$

This algebraic equation has one solution $u(x) = u_0$ which we dismiss as being incompatible with the boundary condition at plus infinity and another solution

$$u(x) = \frac{1}{4} u_0 = u_1 \tag{11.23}$$

Returning to (11.20), we also find

$$\begin{aligned} \rho &= 4(\rho_0 - \rho_\delta) \\ p &= \frac{3}{4} (\rho_0 - \rho_\delta)u_0^2 \end{aligned} \tag{11.24}$$

In particular,

$$RT = \frac{p}{\rho} = \frac{3}{16} u_0^2 = RT_1 \tag{11.25}$$

Both u and T are constant throughout the flow, leaving $\rho(x)$ as the only nontrivial moment. This is the entire information to be gleaned from the moment equations, but there

remains Eq. (11.13) for $\rho_\delta$. The moment $\nu_0$ of f is a function of temperature and is proportional to $\rho$, $\nu_0 = \alpha(T)\rho$. Since $T = T_1$ is constant, we can write

$$\nu_0 = \alpha_1 \rho \tag{11.26}$$

where $\alpha_1$ is a constant, and, between (11.13) and (11.24) obtain

$$\frac{d\rho_\delta}{dx} = -\frac{4\alpha_1}{u_0}(\rho_0 - \rho_\delta)\rho_\delta \tag{11.27}$$

This has the solution

$$\rho_\delta = \frac{\rho_0 e^{-y}}{1 + e^{-y}} \tag{11.28}$$

and

$$\rho = \frac{\rho_1}{1 + e^{-y}} \tag{11.29}$$

($\rho_1 = 4\rho_0$), also

$$\bar{\rho} = \rho + \rho_\delta = \frac{\rho_1 + \rho_0 e^{-y}}{1 + e^{-y}} = \rho_0\left[\frac{5}{2} + \frac{3}{2}\tanh y/2\right] \tag{11.30}$$

where

$$y = x/\hat{\lambda} \tag{11.31}$$

and

$$\hat{\lambda} = \frac{u_0}{\alpha_1 \rho_1} = \frac{u_0}{\hat{\nu}} \tag{11.32}$$

The quantity $\hat{\nu} = \alpha_1 \rho_1$ is the collision frequency of a molecule at speed $u_0$ in a Maxwellian $f_1$; or, it is the collision frequency of a molecule at speed $u_0 - u_1 = 3u_1 = (3RT_1)^{1/2}$ in a rest frame Maxwellian at density $\rho_1$ and temperature $T_1$.

The net velocity $\bar{u}$ is easily obtained from $\bar{\rho}\bar{u} = \rho_0 u_0$,

$$\bar{u} = u_0 \frac{1 + e^{-y}}{4 + e^{-y}} \tag{11.33}$$

The total moments $\bar{p}$ and $R\bar{T}$ are obtained by integrating over $\bar{f} = f + f_\delta$,

$$\bar{p} = p_1 \frac{1 + \frac{1}{2} e^{-y}}{(1 + e^{-y})(1 + \frac{1}{4} e^{-y})}$$

$$\bar{T} = T_1 \frac{1 + \frac{1}{2} e^{-y}}{(1 + \frac{1}{4} e^{-y})^2} \tag{11.34}$$

The $\overline{T}$ profile is flatter at the downstream end than any other variable; the departure from $T_1$ is $e^{-2y}$ instead of $e^{-y}$. This is particularly interesting because a very slight overshoot of $\overline{T}$ is sometimes found at this end [Salwen, et al, 1964]. Our approximation is too crude to decide this question. But the higher order departure of $T_1 - \overline{T}$ indicates that this effect may be quite sensitive to the exact approximation used. Separate parallel and transverse temperatures are sometimes defined; they are

$$T_{\parallel} = T_1 \frac{1 + e^{-y}}{(1 + \frac{1}{4} e^{-y})^2}$$

$$T_{\perp} = \frac{T_1}{1 + \frac{1}{4} e^{-y}} \qquad (11.35)$$

The greatest anisotropy is at the upstream end where $T_{\parallel}/T_{\perp} \sim 4$.

The fact that this very crude approximation yields any answer at all indicates that a moderate improvement, e.g., equivalent to thirteen moments (or a modification such as that of [Butler and Anderson, 1967]) should yield quite accurate results. Investigation of the shock equation (11.11) for the behavior at $x \longrightarrow \pm\infty$ and $\xi \longrightarrow \infty$ might suggest better approximations. Also, if f is unbounded

at $\xi = u_0$ [this appears likely from the term $Q_1(f,f_\delta)$ in (11.11)] a bimodal approximation to f would seem to be called for. For $\bar{f}$ this would involve a sum of three terms, two peaked at $u_0$ with spreads of $T_0$ and $T_1$ and one at $u_1$; this is essentially trimodal.

Our three-moment approximation is closely related to the Mott-Smith bimodal assumption. But the specific relation (11.14) for $\rho$ replaces the more or less arbitrary moment equation which is needed to supplement the Mott-Smith prescription; also, the fact that u and T are constant arises naturally rather than by fiat.

Elaborations of the Mott-Smith approximation (the most accurate is probably [Salwen, et al, 1964]) are found to be consistent with our statement that the correct length scale for a strong shock is the downstream mean free path. It is likely that any of these ad hoc approximations with enough adjustable parameters is fairly accurate. But an equivalent amount of effort is likely to produce much more accurate results if applied to the infinite strength equation (11.11). This has yet to be done.

The separation (11.5) into a "molecular beam" $f_\delta$ plus remainder has been introduced, independently, by I.M. Scholnick and D.L. Turcotte (private communication). The remainder, f, is approximated by using the Navier-Stokes equations. This use of the Navier-Stokes equations

can only be considered as an ad hoc procedure, possibly more accurate than our local Maxwellian, but less easily assessed than, say, a thirteen moment approximation to f. In the present context the Navier-Stokes approximation for second and third moments of f has no intrinsic significance, as it does in a small mean free path expansion.

For the determination of f, it is sufficient to know that $f_\delta$ is concentrated near $\xi = u_0$. But as a separate problem, we can investigate the structure of the distribution $f_\delta$ on the small scale $\xi - u_0 \sim (RT_0)^{1/2}$. For hard potentials (harder than a Maxwellian fifth power), self collisions between $f_\delta$ and itself occur at a much slower rate than $f_\delta$ with f. On the distance scale that $f_\delta$ decays to zero, we can set $Q(f_\delta, f_\delta) = 0$. Equation (11.10) can be solved

$$f_\delta = f_0 \exp\left(-\frac{1}{u_0} \int_{-\infty}^{x} \nu_0 \, dx\right) , \qquad (11.36)$$

and we find that $f_\delta$ remains Maxwellian while it decays.

The fact that the approximate profile (11.30) for an infinite strength shock is a hyperbolic tangent just like the exact weak shock profile, allows us to make a very simple interpolation for the shock thickness that should be accurate to better than a factor two for all strengths.

The Navier-Stokes weak shock formula (11.1) extrapolates to an infinite strength shock thickness ($\varepsilon = 1$) of just $\lambda^*$; (we, of course, use the downstream temperature for $\mu(T)$ to evaluate $\lambda^*$). The equivalent thickness from the present theory is $2\hat{\lambda}$ (11.30). For hard spheres a simple calculation shows that $\hat{\lambda} \sim 0.98\ \lambda^*$. In other words, the infinite strength shock thickness estimated from (11.11) by a local Maxwellian approximation is just twice the weak Navier-Stokes thickness, $\lambda^*$, extrapolated to infinite strength.*

The hyperbolic tangent profile is exact for a weak shock but can only be expected to be a rough approximation for strong shocks. However, the very strong asymmetry that is found in strong shock profiles using a relaxation model for the Boltzmann equation [Liepmann, et al., 1962] seems to be specific to the model itself. The creation term of the Boltzmann equation, $\int f' f_1'$, is replaced by a Maxwellian source at the local temperature, which approximates $T_0 \sim 0$ for the downstream part of the profile. This

*The Navier-Stokes thickness for hard spheres calculated exactly for an infinite strength shock is some 40% smaller than the weak shock extrapolation [Grad, 1952, Fig. 14]. Empirically, it is more accurate (as well as simpler) to extrapolate the weak shock formula than to calculate the correct Navier-Stokes profile.

is quite different from our scaling which states that molecules are created with a velocity spread comparable to $T_1$ for the whole profile. More precisely, the correct upstream limiting values of the moments u and T of the residual distribution f are not known (they turn out to be $u_1$ and $T_1$ in our locally Maxwellian approximation to f). Only if it were to turn out that $u \longrightarrow u_0$ and $T \longrightarrow 0$ for the actual solution at minus infinity would our Ansatz (11.5) be qualitatively comparable with a model equation.

## 12. Conclusion

Taking fluid dynamics as our model, we can look at the theory of the Boltzmann equation as a subject which is several orders of magnitude more extensive, containing an endless variety of boundary layers and non-uniform limits. Enough evidence has been accumulated to indicate that a large part of the subject is accessible to standard methods, particularly asymptotics and boundary layer analysis. But some aspects, possibly including noncutoff potentials, nonself-adjoint spectra, nonlinear existence in the large, may require significantly new approaches.

In trying to project from the past, we can distinguish three qualitatively different eras: transport coefficients, *ad hoc* polynomial and moment methods, and the blossoming of more precise mathematical investigations.

Until 1949 the subject of kinetic theory was essentially coextensive with the Chapman-Enskog theory. The *raison d'être* of the Boltzmann equation was to compute transport coefficients in order to complete the classical equations of continuum mechanics. The phrase "solve the Boltzmann equation" is synonymous, in this era, with the operation of solving the Hilbert integral equation. The concept of the Boltzmann equation as an entity in itself, to be solved subject to initial and boundary conditions was missing

(except for Carleman, whose singular relation to the subject was only revealed posthumously [Carleman, 1957]).

The break with this preoccupation with transport coefficients was made by the introduction of the concept of interpolating different levels of description of a gas [Grad, 1949] between the (at that time) almost inaccessible Boltzmann description via a distribution function and the macroscopic description via fluid moments through the Chapman-Enskog hierarchy. Polynomial and moment approximations gave an ad hoc scheme for this interpolation.

In a few specific problems, viz. linearized and one-dimensional, high order polynomial expansions were carried out [Wang Chang and Uhlenbeck, 1952 and 1953; Mott-Smith, 1954; Pekeris, et al., 1962]. More accurate (but more special and more complicated) methods of parameter fitting were the introduction of half range polynomial expansions [Gross and Ziering, 1958; Lees, 1959] and bimodal approximations of varying degrees of complexity [Mott-Smith, 1951; Salwen, Grosch and Ziering, 1964]. Quite general schemes have been introduced which perturb by polynomial expansion a suitable weight function chosen to fit the problem [Kahn and Mintzer, 1965; Butler and Anderson, 1967]. All these techniques are essentially

ad hoc, and the ratio of accuracy to effort (convergence is unknown) is a sensitive function of the cleverness of the performer.

Mathematical precision has developed more recently in two directions: in general estimates of convergence (i.e., in existence theory) and in precise estimates of the behavior of solutions in special problems and singular limits.

The text of this paper serves as a summary of the increased precision which has developed in special problems and singular limits.

The early existence theory was confined to spatially homogeneous problems [Carleman, 1933; Wild, 1951; Morgenstern, 1954] or gave estimates which became intolerable after one or two collision times [Carleman, 1957; Grad, 1958] or one or two mean free paths [Willis, 1961]. More recently a strong existence theory has been developed for the linearized initial value problem [Grad, 1963b]. Weaker but very promising results have been obtained in boundary value problems [Cercignani, private communication], and a start has been made toward a nonlinear theory [Grad, 1965; Pao, private communication].

The judicious application of ad hoc parameter fitting directed by the growing supply of precise information about singularities offers the most promising hope for accurate practical calculations. We have already pointed out the application to the fundamental solution and the strong shock wave. There are endless variations, such as to extend the precise result for high frequency sound [Grad, 1966] to lower frequencies where it may overlap the normal mode analysis from the other direction [Wang Chang and Uhlenbeck, 1952; Pekeris, et al., 1962; Sirovich and Thurber, 1963]; or to modify a half range expansion to incorporate the velocity boundary layer at $\xi = 0$.

Extrapolating towards the future, we submit the following as a suggestion with regard to the most promising directions and most urgent needs:

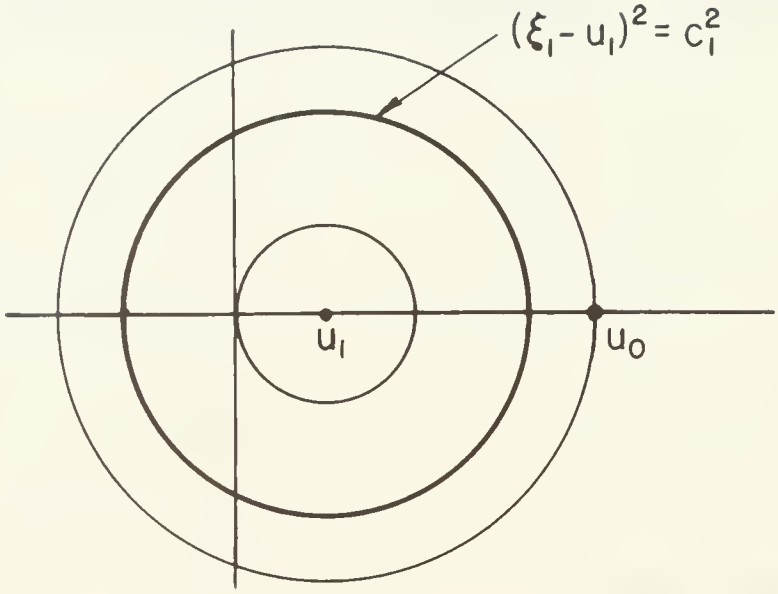
1) stronger existence theorems for boundary value problems to try to bring the level up to what is known for the initial value problem;

2) further exploitation of the abundance of known singular and nonuniform limits;

3) development of numerical techniques to be able to make use of the many explicit answers (in near free flow) which are given as multidimensional integrals, and to serve as numerical experiments for comparison with theory;

4) study of infinite range potentials (which is the real goal of all kinetic investigations), either to discover properties of the collision operator or to solve specific elementary flows.

It is safe to say that kinetic theory for monatomic rarefied gases has reached a stage where the obstructions are mathematical techniques rather than physical principles.


$(\xi_1 - u_1)^2 = c_1^2$
$u_1$
$u_0$

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